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THE FALL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The seventeenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Washington and Jefferson College, Washington, Pennsylvania, on Saturday, October 25, 1941. Professor J. S. Taylor, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was fifty-two, including the following twenty-six members of the Association: O. F. H. Bert, J. O. Blumberg, W. G. Brady, A. M. Bryson, W. E. Buker, A. B. Cunningham, H. A. Davis, L. L. Dines, H. L. Dorwart, Mary B. Ferguson, H. C. Hicks, R. P. Johnson, M. L. Manning, David Moskovitz, L. T. Moston, F. W. Owens, Helen B. Owens, C. N. Reynolds, J. B. Rosenbach, H. C. Shaub, J. S. Taylor, R. W. Thomas, C. H. Vehse, M. L. Vest, W. J. Wagner, E. A. Whitman.

The following officers were elected for the coming year: Chairman, R. G. Sturm, Aluminum Research Laboratories, New Kensington, Pennsylvania; Secretary, H. L. Dorwart, Washington and Jefferson College; member of Executive Committee, E. D. Wells, Erie Center, University of Pittsburgh. Professor C. H. Vehse, West Virginia University, continues in office for the second year of his term as the additional member of the Executive Committee.

At the close of the afternoon session, tea was served for the guests. A vote of thanks was extended to the staff of the College for their generous hospitality.

After an address of welcome by Professor O. F. H. Bert of Washington and Jefferson College, the following six papers were read:

1. "Beyond quadratics" by Professor H. L. Dorwart, Washington and Jefferson College.

2. "History of the cycloid" by Professor E. A. Whitman, Carnegie Institute of Technology.

3. "A report on the aims and activities of the committee on the teaching of science in the secondary schools" by Professor W. H. Michener, Carnegie Institute of Technology, introduced by the Secretary.

4. "A non-involutorial space transformation associated with a $Q_{1,n}$ congruence" by M. L. Vest, West Virginia University.

5. "Fermat's last theorem" by E. C. Johnson, Somerset, Pennsylvania, introduced by Professor Taylor.

6. "A report on the summer program in mechanics at Brown University" by A. M. Bryson, University of Pittsburgh.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Dorwart gave an elementary account of the origin and nature of modern abstract algebra. The paper is to be published in the *National Mathematics Magazine*.

2. Professor Whitman traced briefly the history of the cycloid from the time of Galileo's attempts (1599) to find the area under one arch to the proof by Jean Bernoulli (1696) that this curve is also a brachistochrone. Roberval's method of finding the area was shown in detail, and comments were made on the discovery of three ways of finding tangents and also on Huygens's method of finding the length of an arch. Professor Whitman stated that he found this history was always interesting to students in the calculus and also stimulating in that it showed the power of the calculus as compared to the methods it displaced.

4. Mr. Vest showed that the transformation, of order $m+m'+n$, is associated with the congruence of lines on a plane curve r of order n having an $(n-1)$-point and a secant s through the multiple point. Two projective pencils of surfaces $|F|$ and $|F'|$ of order m and m' contain the secant as $(m-1)$-fold and $(m'-1)$-fold line, respectively. Through a generic point $P(y)$ there passes a single surface F of $|F|$. The unique line t through $P(y)$, s and r meets the associated F' of $|F'|$ in one residual point $P'(x)$, the image of $P(y)$ under the transformation thus defined.

5. Mr. Johnson, an amateur mathematician, explained how he has developed an original approach to the solution of Fermat's last theorem, which requires a proof that $x^n+y^n=z^n$ cannot be satisfied by whole numbers when n is a whole number greater than 2. Through a comprehensive study of the theory of the algebraic form a^2+ab+b^2, which he has published in book form, he has demonstrated the propriety of Euler's assumptions. In addition to his published work.

Mr. Johnson has among his unpublished work a proof that $x^{p-1}+y^{p-1}=z^{p-1}$ cannot be satisfied by whole numbers when p is any prime number greater than 3. This, he believes, is the nearest approach which has been made to a complete solution of Fermat's last theorem.

6. Mr. Bryson gave a survey of the courses of lectures which he attended at Brown University during the summer of 1941.

DAVID MOSKOVITZ, *Secretary*

HEAT FLOW AND NON-EUCLIDEAN GEOMETRY

E. W. BARANKIN, University of California

In *La Science et l'Hypothèse* (Paris, Ernest Flammarion) Poincaré has well established a very fundamental property of physical space—its *passivity*; that is, the applicability thereto of an arbitrary geometry. There, on pp. 84 to 87, he illustrates the thesis with the example of a 3-dimensional world consisting of the interior of a sphere of radius R, each concentric sphere of radius $r<R$ being held at a constant absolute temperature $T \propto R^2-r^2$, and objects in this world being subject, under displacement, to expansion and contraction accordingly. We propose to develop here a similar example of a 2-dimensional world, at once treating rather thoroughly its purely mathematical side and emphasizing its physical import. Concerning the latter, the significant principle which will be brought out is that of the functional relationship between the geometry of physical space and physical law.

Let us imagine that we one day find it necessary to deal in the laboratory with a rather large sheet of copper. The sheet is not, however, of uniform overall temperature; on the contrary, if we take one edge of the sheet to be the x-axis, and erect perpendicular to it a y-axis, then the temperature, in degrees Centigrade, is the following function of the Euclidean coördinates:

$$T = b\left(y - \frac{1}{pb}\right),$$

where b and p are constants. For the present, let there be constraints on the sheet to prevent it from expanding or buckling. Finally, we are in possession of an infinitesimal measuring rod of iron, whereof we ideally take the thermal coefficient of expansion to be p for all T; and we must meet the problem of establishing geometric relations on the sheet.

We should at first be very reluctant to measure distances on the sheet with the iron rod, knowing that it would alter in length as we moved it from place to place.* Rather would we ask for a rod of invar, a metal with a coefficient of

* For we shall agree to make measurements in a region only after the rod has come to thermal equilibrium with that region.

expansion of zero. But in the event that the iron rod were the only one available, we should have to resign ourselves to its use. And the question arises: how can we be consistent in our measurements with such a rod?

There are, in fact, two simple ways:

1. At 0° (the standard temperature at which we take all true or Euclidean length-measures) let the rod be of true length ds; its reading then at all positions on the sheet is ds. But the true length at the point (x, y) is:

$$ds' = [1 + pT]ds = pbyds.$$

Hence, we may preserve Euclidean measure, and therefore Euclidean geometry, for the surface if we carry the temperature correction factor (pby) over the surface. In this case the temperature correction factor is the physical law pertaining to the matter in the space, and the corresponding geometry is Euclidean.

2. *Perform measurements without regard to the changes in the rod.* This suggestion is at first glance rather shocking; and it is not immediately obvious that such a course is a consistent one. Here we are taking as the governing physical law: matter in the space undergoes no changes. Clearly this law is simpler than that of case 1; it remains to discover whether or not there corresponds a permissible geometry. We occupy the following pages with a determination of this geometry and a demonstration of its consistency.

In the first place, it is seen that the x-axis is the line at infinity; for if the length of a line segment perpendicular to the x-axis and having its foot on this axis is measured with a rod, the rod will become shorter and shorter; and in the neighborhood of the axis the rod must repeat itself a great number of times to measure a very small true-length. In other words, this segment will, when measured toward the x-axis, appear infinite in length.

We now turn to the problem, foremost in importance, of discovering in what ways we may move a physical object, such as a triangle* made up of rods, about in the space so that the object remains congruent to itself. Such a movement we call a *displacement*. Obviously all displacements are sense-preserving. But the notion of anticongruence may well figure in our space in the form of a sense-reversing transformation. We therefore employ the broader term *motion* for any congruence transformation. We may then set ourselves the task of determining the set of motions of our space; the set of displacements will be just the collection of sense-preserving motions.

This problem is readily attacked with infinitesimal considerations; but first we must derive the metric of the space. At a point (x, y) we shall have, if the Euclidean projections of the rod on the x- and y-axes are, respectively, dx and dy, the following:

$$pbyds = [dx^2 + dy^2]^{1/2},$$

* In general, a triangle is defined as a closed, three-sided figure, the edges being segments of geodesics.

whence immediately we have our metric:

$$ds^2 = \frac{dx^2 + dy^2}{(pb)^2 y^2}.$$

We may choose our units in such a way that pb is unity, and write:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{1}$$

We emphasize here the fact that the Euclidean coördinates which we have called up constitute merely a frame for analytical treatment, and do not participate intrinsically in the geometry to be developed.

In terms of the metric, a motion is a transformation,

$$x' = x'(x, y), \qquad y' = y'(x, y), \tag{2}$$

to be thought of as a point transformation referred to the above invariable cartesian axes—such that:

$$\frac{dx'^2 + dy'^2}{y'^2} = \frac{dx^2 + dy^2}{y^2}. \tag{3}$$

That this is so follows from a brief consideration. First, equation (3) obviously guarantees the preservation of distance. It also implies the preservation of angles (conformality), in giving:

$$\begin{aligned} \frac{1}{y'^2}\left[\left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial y'}{\partial x}\right)^2\right] &= \frac{1}{y^2}, \\ \left[\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y}\right] &= 0, \\ \frac{1}{y'^2}\left[\left(\frac{\partial x'}{\partial y}\right)^2 + \left(\frac{\partial y'}{\partial y}\right)^2\right] &= \frac{1}{y^2}. \end{aligned} \tag{4}$$

For, let (d_1x, d_1y) and (d_2x, d_2y) be two infinitesimal segments, drawn from the point (x, y) and there making an angle θ. Further, let (d_1x', d_1y') and (d_2x', d_2y') be their respective transforms under (2). θ is given by:

$$\begin{aligned} \cos\theta &= \frac{d_1x d_2x + d_1y d_2y}{[(d_1x^2 + d_1y^2)(d_2x^2 + d_2y^2)]^{1/2}} \\ &= \frac{\dfrac{1}{y^2}(d_1x d_2x + d_1y d_2y)}{\left[\left(\dfrac{d_1x^2 + d_1y^2}{y^2}\right)\left(\dfrac{d_2x^2 + d_2y^2}{y^2}\right)\right]^{1/2}}. \end{aligned} \tag{5}$$

Likewise, the angle θ' formed by the transformed segments at (x', y') is given by:

$$\cos\theta' = \frac{\frac{1}{y'^2}(d_1x'd_2x' + d_1y'd_2y')}{\left[\left(\frac{d_1x'^2 + d_1y'^2}{y'^2}\right)\left(\frac{d_2x'^2 + d_2y'^2}{y'^2}\right)\right]^{1/2}}. \tag{6}$$

Under (3) the denominators of (5) and (6) are equal and we have the condition for conformality:

$$\frac{1}{y'^2}(d_1x'd_2x' + d_1y'd_2y') = \frac{1}{y^2}(d_1xd_2x + d_1yd_2y). \tag{7}$$

If we apply to the left-hand member of (7) the induced transformation,

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy, \qquad dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y}{\partial y}dy,$$

and bring to bear the relations (4), we find that the left-hand member reduces indentically to the right-hand member, wherefore the validity of our analytic characterization of a motion is fully established.

Take the infinitesimal line segment of Euclidean length δs (Fig. 1) and any point P_0 on the x-axis; describe rays from P_0 through the end-points of δs. Then

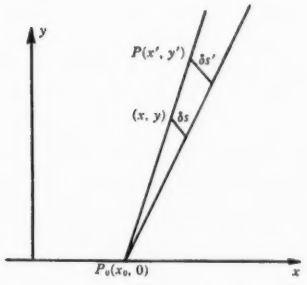

FIG. 1

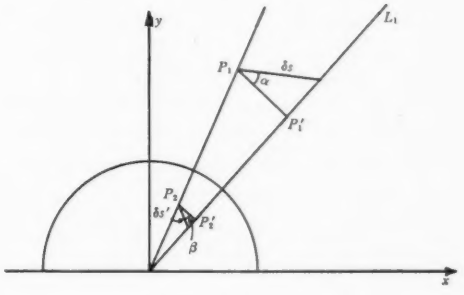

FIG. 2

at any arbitrary point P on one of the rays draw the segment of Euclidean length $\delta s'$ parallel to δs and ending on the second ray. The segment $\delta s'$ is equal to the segment δs under our non-Euclidean metric. For, by similar triangles,

$$\frac{\delta s}{y} = \frac{\delta s'}{y'}.$$

But the non-Euclidean length of the segment δs is given by (1) as

$$ds = \frac{\delta s}{y}$$

and that of the segment $\delta s'$ as:

$$ds' = \frac{\delta s'}{y'}.$$

Hence:

$$ds' = ds.$$

This equation is identifiable with (3), the pertinent transformation being:

$$x' = ax + (1 - a)x_0, \qquad y' = ay, \qquad (a \geqq 0), \tag{8}$$

where a is the factor of magnification.

A second motion makes itself evident when we observe the form of the metric closely. Since only the differential of the variable x occurs, and since the differential is invariant under the addition of a constant to the variable, it is immediately clear that we obtain a motion by augmenting x by a constant and leaving y unchanged. This amounts to the translation parallel to the x-axis:

$$x' = x + t, \qquad y' = y, \tag{9}$$

where t is the parameter of translation.

We know that if, in a two-dimensional space, there definitely exist two independent motions, then a third independent motion must exist.* We therefore seek the third motion; and in fact, we discover this to be an inversion. For, consider again the infinitesimal segment of Euclidean length δs at P_1, and the unit circle (Fig. 2). Inversion in this circle carries δs into $\delta s'$. Construct the segments $\overline{P_1P_1'}$ and $\overline{P_2P_2'}$ perpendicular to the ray L_1. Then, as proved above, segment $\overline{P_1P_1'}$ is equal to segment $\overline{P_2P_2'}$ under the metric (1). Moreover, inversion being a conformal transformation, we have $\alpha = \beta$. It follows therefore, since

$$\cos \alpha = \frac{\overline{P_1P_1'}}{\delta s} = \cos \beta = \frac{\overline{P_2P_2'}}{\delta s'},$$

that segment δs is equal to segment $\delta s'$ under (1). Analytically this inversion is:

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2}. \tag{10}$$

---

* See Eisenhart, Riemannian Geometry, p. 241, par. 74.

A similar derivation could have been given for any circle centered on the x-axis, for example, that of radius r with center at $(x_0, 0)$. The resultant inversion, $I_{x_0,r}$,

$$x' = x_0 + \frac{r^2(x - x_0)}{(x - x_0)^2 + y^2}, \qquad y' = \frac{r^2 y}{(x - x_0)^2 + y^2},$$

were then a product of the above simple motions in the following order (Fig. 3): First a translation T_1 of δs:

$$x' = x - x_0, \qquad y' = y,$$

then an inversion $I_{0,1}$ in the unit circle:

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2},$$

a magnification, M, through the origin:

$$x' = r^2 x, \qquad y' = r^2 y,$$

and finally a translation T_2:

$$x' = x + x_0, \qquad y' = y.$$

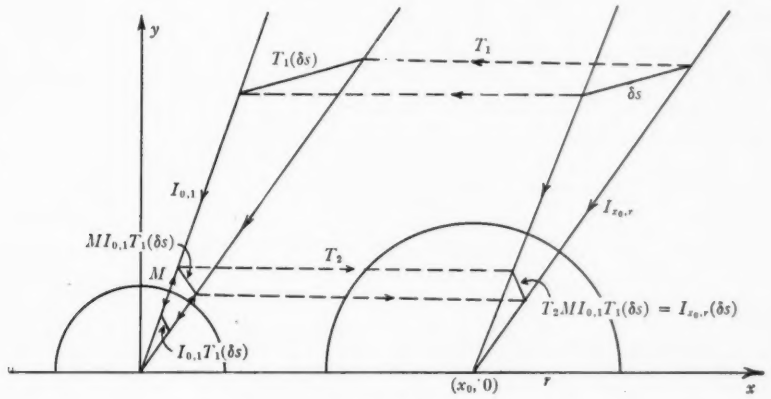

FIG. 3

Concerning the determination of these component transformations, we may remark that the translation T_1 is first performed to bring the domain of the inversion $I_{x_0,r}$ into radial coincidence with that of $I_{0,1}$, the magnification, to account for the different radii of the two circles of inversion, and the translation $T_2 = T_1^{-1}$, to return the domain of $I_{x_0,r}$ to its original position.

The motion of inversion introduces the notion of anti-congruence into our space. In this connection it is of interest to note that Euclid, in order to prove

completely his propositions on the congruence of triangles, had to make use of superposition, an exaction on the third dimension. This difficulty arises when the three independent motions of the Euclidean plane are taken as two translations and a rotation, in the stead of two translations and a reflection. Of course, a rotation may be obtained by the multiplication of two reflections.

We now have the complete* set of three basic independent motions, a translation parallel to the x-axis, a magnification through the origin, and an inversion in the unit circle. These motions forms a group G† which is called the congruence group of the space. What further information we require concerning this group can be most easily obtained by expressing the basic motions in terms of the complex variables $z=x+iy$ and $w=x'+iy'$. We have:

$$
\begin{aligned}
&\text{Translation,} && T: && w = z + t, && (t, \text{ real}),\\
&\text{Magnification,} && M: && w = az, && (a \geqq 0),\\
&\text{Inversion,} && I: && w = 1/\bar{z}.
\end{aligned}
\tag{11}
$$

The presence of *three* independent motions in the group is the abstract equivalent of the fact that a two dimensional configuration in the space may be shifted arbitrarily and, so to speak, turned inside-out and still remain congruent, or in the latter case, anticongruent, to itself. This complete freedom of movement is precisely what we hoped to discover for the case of a physical object, and we shall therefore establish the fact explicitly for sense-preserving motions, that is, displacements.

The set of displacements forms a subgroup $\overline{G}$ of G; and $\overline{G}$ is generated by the three independent, sense-preserving motions:

$$
\begin{aligned}
&T: && w = z + t,\\
&M: && w = az,\\
&U: && w = \frac{z}{fz + 1}, && (f, \text{ real}).
\end{aligned}
\tag{12}
$$

The latter claim requires a detailed proof: A displacement is any product of the fundamental motions (11) involving an even number of inversions. It is easily shown that such a general product takes the form D:

$$w = \frac{kz + l}{mz + n}$$

where k, l, m, n are real numbers and $kn - ml \geqq 0$. But this general displacement

* In an N-dimensional space there are at most $N(N+1)/2$ independent motions.

† This fact is easily proved; we shall not perform the work here.

is obtained by the multiplication of the following displacements of the forms (12):

$$
\begin{aligned}
U_1\colon &\quad w = \frac{z}{\frac{m}{n}z + 1}, \\
M_1\colon &\quad w = \left(\frac{kn - ml}{n^2}\right)z, \\
T_1\colon &\quad w = z + \frac{l}{n}.
\end{aligned}
\tag{13}
$$

In fact, $D = T_1 M_1 U_1$.

The displacement U is the product of an inversion in the unit circle, a translation by amount f, and another inversion. Our method of replacing the inversion of the basis (11) by U, to obtain a basis for the group $\overline{G}$, is equivalent to the substitution, in the case of the Euclidean plane, of a rotation (the product of two reflections) for the basic motion of reflection to obtain a basis for the group of Euclidean displacements.

Now, an arbitrary shifting of a physical object in our space may be characterized in this manner: the allocation of a particular point of the object with an arbitrarily chosen point of the space, and the alignment of a particular direction fixed in the object with an arbitrarily chosen direction in the space. This process clearly requires only *three* parameters for its completion, and the group $\overline{G}$ is a three-parameter group. It is therefore proved that an object in the space enjoys free mobility, a fact denoted by calling the space a *congruence space* and its geometry, a *congruence geometry*.

We now inquire after the geodesics of our space. For the distance between two points P_1 and P_2 we have

$$
s = \int_{P_1}^{P_2} \frac{1}{y}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{14}
$$

If $F \equiv 1/y\sqrt{1+(dy/dx)^2}$, the Euler equation for extremality of this integral is

$$
\frac{d}{dx}\frac{\partial F}{\partial\left(\frac{dy}{dx}\right)} = \frac{\partial F}{\partial y}.
$$

We are led, by the substitution for F, to the differential equation of the family of geodesics,

$$
\frac{d^2y}{dx^2} + \frac{1}{1+y}\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0.
$$

The solution of this equation gives the family of circles:

$$(x - h)^2 + y^2 = c^2, \tag{15}$$

where h and c are constants of integration. All these circles are centered on the x-axis, and the upper half-circumferences are the geodesics of our space. The geodesic through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with different ordinates is seen to degenerate into a straight line parallel to the y-axis when x_2 approaches x_1 in value.

In the Euclidean plane there can be drawn through a given point one and only one straight line parallel to a given straight line. Or, in general terms, there can be drawn through a given point one and only one geodesic parallel to a given geodesic, that is, not intersecting it in any finite point. This is not the case in the heated plane. For, let C_1 be a given geodesic; and let P be a point not on C_1. Describe the geodesics C_2 and C_3 tangent to C_1 at the points where it meets the x-axis. Then both C_2 and C_3 are parallel, in the sense defined above, to C_1. Moreover any geodesic whose slope at P lies between those of C_2 and C_3 is also parallel to C_1.

The student of the non-Euclidean geometries will have immediately recalled the name Lobatschewsky when he read the above result concerning parallels. For it is so in the Lobatschewskian geometry that to a given straight line (geodesic) can be drawn a limited infinity of parallels through a neighboring point.* And in fact our heated-plane geometry is exactly that of Lobatschewsky. A little exercise with triangles in this plane will soon convince the reader that the sum of the angles in a triangle is always less than two right angles.

Two further computations bear results of interest from the projective point of view. First we compute the geodesic distance s_g between two points. Let the two points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$; then, substituting $\sqrt[+]{c^2-(x-h)^2}$ for y in formula (14), we have:

$$s_g = \int_{x_1}^{x_2} \frac{c}{c^2 - (x - h)^2}\, dx = \tfrac{1}{2} \log_e \left\{ \frac{x_2 - (h - c)}{(h + c) - x_2} \cdot \frac{(h + c) - x_1}{x_1 - (h - c)} \right\}.$$

But, if x_3 and x_4 ($x_4 > x_3$) denote respectively the x-coördinates of the two points in which the geodesic meets the x-axis, we find by (15) that

$$x_3 = h - c, \qquad x_4 = h + c.$$

Hence:

$$s_g = \tfrac{1}{2} \log_e \left\{ \frac{x_2 - x_3}{x_4 - x_2} \cdot \frac{x_4 - x_1}{x_1 - x_3} \right\},$$

* Some mathematicians choose to say that there are only two parallels to a given geodesic through a point, in this geometry, corresponding to C_2 and C_3 in the above description; they employ the term *skew* for all the others that we have called parallel. But this is purely a matter of definition.

or, in terms of the cross-ratio

$$R_1 = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_2 - x_4)(x_1 - x_3)}$$

of the four ordered points $(x_1, 0)$, $(x_2, 0)$, $(x_4, 0)$, $(x_3, 0)$,

$$s_g = \tfrac{1}{2} \log_e R_1.$$

Secondly, we derive the formula for the angle between two geodesics. Let C_1 and C_2 be two geodesics, intersecting at the point (x_0, y_0), and there making an angle θ. From equation (15) we derive the respective slopes of the two geodesics:

$$\mu_1 = -\frac{x - h_1}{y}, \qquad \mu_2 = -\frac{x - h_2}{y}.$$

Hence,

$$\tan\theta = \left(\frac{\mu_2 - \mu_1}{1 + \mu_2\mu_1}\right)_{(x_0, y_0)} = \frac{\dfrac{h_2 - h_1}{y_0}}{1 + \dfrac{(x_0 - h_1)(x_0 - h_2)}{y_0^2}}.$$

If, for $i = 1, 2$, we let the geodesic, C_i intersect the x-axis in the points $(x_{i1}, 0)$, and $(x_{i2}, 0)$ with $x_{i1} > x_{i2}$, we have

$$y_0^2 = (x_0 - x_{11})(x_{12} - x_0) = (x_0 - x_{21})(x_{22} - x_0).$$

The last two members give:

$$x_0 = \frac{x_{21}x_{22} - x_{11}x_{12}}{(x_{21} + x_{22}) - (x_{11} + x_{12})}$$

which in turn gives:

$$y_0 = \frac{\sqrt{(x_{22} - x_{11})(x_{22} - x_{12})(x_{21} - x_{11})(x_{12} - x_{21})}}{[(x_{21} + x_{22}) - (x_{11} + x_{12})]}.$$

Substituting these values and the values

$$h_j = \frac{x_{j1} + x_{j2}}{2}, \qquad (j = 1, 2),$$

into the expression for $\tan\theta$, we obtain, after rearrangement,

$$\tan\theta = \frac{2\sqrt{(x_{22} - x_{11})(x_{22} - x_{12})(x_{21} - x_{11})(x_{12} - x_{21})}}{(x_{12} - x_{21})(x_{22} - x_{11}) - (x_{22} - x_{12})(x_{21} - x_{11})}.$$

In terms of the cross-ratio

$$R_2 = \frac{(x_{21} - x_{11})(x_{22} - x_{12})}{(x_{22} - x_{11})(x_{21} - x_{12})}$$

of the four ordered points $(x_{21}, 0)$, $(x_{22}, 0)$, $(x_{11}, 0)$ and $(x_{12}, 0)$, this formula reduces to:

$$\tan^2 \frac{\theta}{2} = -R_2.$$

We have now quite sufficiently established the validity of our second suggested method of treatment of the copper sheet. For, above all, the prime question is settled: the geometry generated by the simpler physical law there laid down is a congruence geometry. And without doubt we should unanimously choose this second method for dealing with the heated sheet, for the new geometry exactly equals that of Euclid in completeness, and we gain in simplicity of physical law.

If the constraints, which we applied in the beginning, were removed from the copper sheet, it would expand in accordance with the varying temperature and buckle into a surface with the metric,

$$ds^2 = y^2(dx^2 + dy^2). \tag{16}$$

To see this, consider, at the point (x, y) on the constrained sheet, a small scratch with components of length dx and dy. When the sheet is allowed to buckle, these components assume new lengths $dx' = dx(1+pT) = ydx$ and $dy' = dy(1+pT) = ydy$, respectively. The expanded length ds of the scratch is then given by

$$ds^2 = dx'^2 + dy'^2 = y^2(dx^2 + dy^2).$$

For working with this surface we should choose a rod of copper in preference to an invar rod. For now the copper rod would expand at a point of the surface in the same way that the sheet itself has expanded at that point, and under the law that the rod suffers no change we should obtain Euclidean geometry for the surface. The invar rod would, under the same simple law, give us the intrinsic geometry involved in the metric (16). But this geometry is *not* a congruence geometry; objects on this surface must, if they are to remain self-congruent throughout the movement, be moved only along the curves $y = \text{constant}$.

FOUR FINITE GEOMETRIES*

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1. Introduction. A finite geometry is a geometry based on a set of postulates, undefined terms, and undefined relations which limits the set of all points and lines to a finite number. This is usually accomplished by a postulate limiting the number of points on a line. In the first three of the finite geometries considered in this paper there is the following postulate: "No line contains more than three points." In the Desargues finite geometry this is a theorem which follows from the postulates. The set of postulates should fulfill the three requirements of consistency, independence, and categoricalness.

Finite geometries were brought into prominence by the publication of the Veblen and Young *Projective Geometry* (Ginn and Co., Vol. I, 1910; Vol. II, 1918) and by Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (The Macmillan Co., 1911).

The simplest example is the finite geometry of 7 points and 7 lines given in Volume I, Chapter I of the *Projective Geometry* of Veblen and Young. This finite geometry was first considered by Fano in 1892 in 3 dimensions where there are 15 points and 35 lines, but in each plane there are 7 points and 7 lines.

The notion of a class of objects is fundamental in logic. The objects which make up a class are called the elements of the class. The notion of a class and the relation "belonging to a class" will be undefined. Given a set S with elements $A_1, A_2, A_3, \cdots$, let S have certain undefined sub-classes any one of which will be called an m-class, —or in particular, given a set of points $A_1, A_2, A_3, \cdots$, let certain sets of points be associated in an undefined way in sets called lines.

2. The seven point finite geometry. The postulates for the 7 point finite geometry may be stated as follows:

(1′). If A_1 and A_2 are distinct points (elements of S), there is at least one line (m-class) containing A_1 and A_2.

(2′). If A_1 and A_2 are distinct points (elements of S), there is not more than one line (m-class) containing A_1 and A_2.

(3′). Any two lines (m-classes) have at least one point (element of S) in common.

(4′). There exists at least one line (m-class).

(5′). Every line (m-class) contains at least three points (elements of S).

(6′). All the points (elements of S) do not belong to the same line (m-class).

(7′). No line (m-class) contains more than three points (elements of S).

Symbolic diagram where the vertical columns represent lines (m-classes):

$$A_1A_2A_3A_4A_5A_6A_7$$
$$A_2A_3A_4A_5A_6A_7A_1$$
$$A_4A_5A_6A_7A_1A_2A_3$$

It is, however, unfortunate that it is necessary to assume that the three points

* Presented at the organization meeting of the Metropolitan New York Section of the Mathematical Association of America at Queens College of the City of New York on April 19, 1941.

A_1, A_3, and A_7, *i.e.*, the diagonal points of the complete quadrangle, are collinear as indicated by the dotted line.

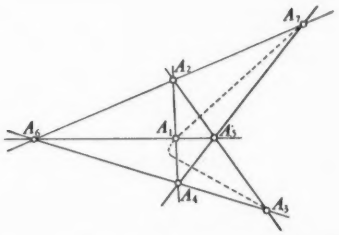


FIG. 1. Geometric diagram for 7 point finite geometry.

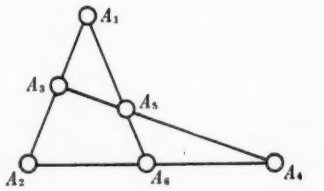
To prove that a postulate of a set of postulates is independent of the rest, it is sufficient to give an example which violates that postulate and fulfills all the rest. Since there are seven postulates in this set, it is necessary to give seven examples to complete the independence proof. This is not always easy to do, and in the case of Hilbert's postulates twenty-one examples would be necessary to complete the independence proof.

If the word "three" were changed to "two" in postulates 5 and 7, the entire geometry would consist of a single triangle which might well be considered as the simplest non-trivial finite geometry; so that seven independent postulates in this case define a geometry consisting of just one triangle.

The independence of the postulates for this finite geometry is shown by the following examples, whose numbers correspond to the number of the postulate which does not hold in the example.

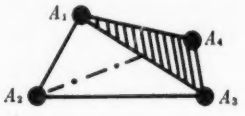
(1′). A complete quadrilateral.

$A_1A_1A_2A_3$
$A_2A_5A_4A_4$
$A_3A_6A_6A_5$



(2′). A tetrahedron $A_1A_2A_3A_4$, where the faces represent lines.

$A_1A_1A_1A_2$
$A_2A_2A_3A_3$
$A_3A_4A_4A_4$



(3′). $A_1A_1A_1A_1A_2A_2A_2A_3A_3A_3A_4A_7$
$A_2A_4A_5A_6A_5A_4A_6A_4A_5A_6A_5A_8$
$A_3A_7A_9A_8A_8A_9A_7A_8A_7A_9A_6A_9$

(This is the Young finite geometry of 9 points and 12 lines, section 3.)

(4′). A single point (the remaining postulates are fulfilled vacuously).

(5′). A triangle with A_1, A_2, A_3 as vertices.

(6′). A single line containing 3 points A_1, A_2, A_3.

(7′). Projective geometry.

This seven point finite geometry has been generalized to give a finite geometry of thirteen points and thirteen lines if four is substituted for three in postulates 5 and 7; and to n^2+n+1 points and lines, if $n+1$ is substituted for three in postulates 5 and 7.

Finite geometries of this type have been treated extensively from the standpoint of algebra and finite groups.*

The question now arises as to what theorems there are in this seven point finite geometry. In the first place, the duals of the postulates may be proved as theorems and the geometry will then have duality. Postulates 1 and 3 are duals.

THEOREM 1. (Dual of Postulate 2). *Two distinct lines have only one point in common.*

THEOREM 2. (Dual of Postulate 4). *There exists at least one point.*

THEOREM 3. (Dual of Postulate 5). *At least three lines pass through every point.*

THEOREM 4. (Dual of Postulate 6). *All lines do not pass through the same point.*

THEOREM 5. (Dual of Postulate 7). *Not more than three lines pass through every point.*

These theorems are all easy to prove, and they show that the geometry has duality. Two other theorems suggest themselves.

THEOREM 6. *The geometry contains precisely seven points.*

THEOREM 7. (Dual of Theorem 6). *The geometry contains precisely seven lines.*

The entire body of theorems of this finite geometry of seven points and seven lines consists primarily of these seven theorems. The finite geometry has the characteristics of a projective geometry and might be considered as the simplest type of a projective geometry.

3. A finite geometry of nine points and twelve lines. If in the postulates of section 2, the word "three" is changed to "four" in postulates 5 and 7, the postulates are satisfied by a finite geometry of 13 points and 13 lines. But, if in this geometry one line of four points is omitted, we obtain a geometry of 9 points and 12 lines, which is equivalent to projecting one line to infinity and converting the projective geometry of 13 points and 13 lines without parallel lines into a euclidean geometry of 9 points and 12 lines with parallel lines.

* Veblen and Bussey, Finite projective geometries, Transactions of the American Mathematical Society, vol. 7, 1906, pp. 241–259.

This is in some ways an advantage and a simplification, because in general there is a preference—for historical reasons—for a euclidean geometry in which the parallel postulate is true. This geometry has been used as an example of a complete logical system by Cohen and Nagel in their book, *An Introduction to Logic and Scientific Method* (Harcourt, Brace and Co., 1934).

It is remarkable that the 9 inflection points of a general plane cubic, as far as collinearity properties are concerned, fulfill all of the postulates of this finite geometry.

The following eight postulates define this finite geometry:

(1). If A_1 and A_2 are distinct points (elements of S), there exists one line (m-class) containing A_1 and A_2.

(2). If A_1 and A_2 are distinct points (elements of S), there exists not more than one line (m-class) containing A_1 and A_2.

(3). Given a line a (m-class a) not containing a point A (given element A of S), there exists one line (m-class) containing A and not containing any point of a (element of S belonging to m-class a).

(4). Given a line a (m-class a) not containing a point A (given element of S), there exists not more than one line (m-class) containing A and not containing any point of a (element of S belonging to m-class a).

(5). Every line (m-class) contains at least three points (elements of S).

(6). Not all points (elements of S) are contained by the same line (m-class).

(7). There exists at least one line (m-class).

(8). No line (m-class) contains more than three points (elements of S).

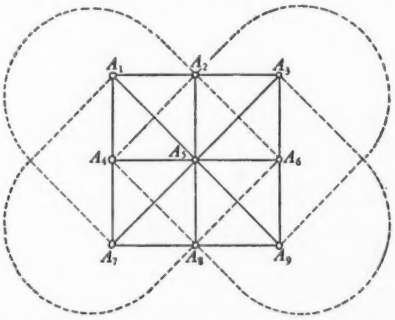


FIG. 2. Geometric diagram for the Young 9 point finite geometry.

Symbolic diagram where the vertical columns represent m-classes (lines):

$$A_1A_1A_1A_1A_2A_2A_2A_3A_3A_3A_4A_7$$
$$A_2A_4A_5A_6A_5A_4A_6A_4A_5A_6A_5A_8$$
$$A_3A_7A_9A_8A_8A_9A_7A_8A_7A_9A_6A_9$$

The independence of the postulates is shown by the following examples,*

* See article by A. Barshop, Brooklyn College Mathematics Mirror, Issue no. VII, 1939, p. 14.

where parentheses represent lines or m-classes.

(1′). Two lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

(2′). Six points (elements) $A_1, A_2, A_3, A_4, A_5, A_6$ taken three at a time to form twenty lines (m-classes).

$$(A_1A_2A_3),\ (A_1A_2A_4),\ (A_1A_2A_5),\ (A_1A_2A_6),\ (A_1A_3A_4),$$
$$(A_1A_3A_5),\ (A_1A_3A_6),\ (A_1A_4A_5),\ (A_1A_4A_6),\ (A_1A_5A_6),$$
$$(A_2A_3A_4),\ (A_2A_3A_5),\ (A_2A_3A_6),\ (A_2A_4A_5),\ (A_2A_4A_6),$$
$$(A_2A_5A_6),\ (A_3A_4A_5),\ (A_3A_4A_6),\ (A_3A_5A_6),\ (A_4A_5A_6).$$

(This is a complete 6-point in space, where the planes represent lines.)

(3′). $(A_1A_2A_4), (A_2A_3A_5), (A_3A_4A_6), (A_4A_5A_7), (A_5A_6A_1), (A_6A_7A_2), (A_7A_1A_3)$.

(This is the seven point finite geometry of section 2.)

(4′). $(A_1\,A_4\,A_5\,),\ (A_2\,A_5\,A_6\,),\ (A_3\,A_6\,A_7\,),\ (A_4\,A_7\,A_8\,),\ (A_5\,A_8\,A_9\,),$
$(A_6\,A_9\,A_{10}),\ (A_7\,A_{10}A_{11}),\ (A_8\,A_{11}A_{12}),\ (A_9\,A_{12}A_{13}),\ (A_{10}A_{13}A_{14}),$
$(A_{11}A_{14}A_{15}),\ (A_{12}A_{15}A_1\,),\ (A_{13}A_1\,A_2\,),\ (A_{14}A_2\,A_3\,),\ (A_{15}A_3\,A_4\,),$
$(A_1\,A_3\,A_9\,),\ (A_2\,A_4\,A_{10}),\ (A_3\,A_5\,A_{11}),\ (A_4\,A_6\,A_{12}),\ (A_5\,A_7\,A_{13}),$
$(A_6\,A_8\,A_{14}),\ (A_7\,A_9\,A_{15}),\ (A_1\,A_8\,A_{10}),\ (A_2\,A_9\,A_{11}),\ (A_3\,A_{10}A_{12}),$
$(A_4\,A_{11}A_{13}),\ (A_5\,A_{12}A_{14}),\ (A_6\,A_{13}A_{15}),\ (A_1\,A_7\,A_{14}),\ (A_2\,A_8\,A_{15}),$
$(A_1\,A_6\,A_{11}),\ (A_2\,A_7\,A_{12}),\ (A_3\,A_8\,A_{13}),\ (A_4\,A_9\,A_{14}),\ (A_5\,A_{10}A_{15})$.

(5′). A complete quadrilateral.

$$(A_1A_2),\ (A_1A_3),\ (A_1A_4),\ (A_2A_3),\ (A_2A_4),\ (A_3A_4).$$

(6′). A single line of three points $(A_1A_2A_3)$.

(7′). A single point A_1—no lines.

(8′). Plane euclidean geometry.

This finite geometry is euclidean in the sense that through any point not on a line there is one and only one line parallel to that line. The geometry does not have the property of duality because any two distinct points determine one line, but any two distinct lines do not determine a point since they may be parallel.

Several theorems suggest themselves, such as the following:

THEOREM 1. *There exist exactly nine points.*

THEOREM 2. *There exist exactly twelve lines.*

THEOREM 3. *Every line has precisely two lines parallel to it.*

THEOREM 4. *Two lines parallel to a third line are parallel to each other.*

THEOREM 5. *The six points on two parallel lines determine a hexagon such that the intersection points of opposite sides are collinear.* (Pappus-Pascal theorem).

4. The Pappus finite geometry. The postulates of the Pappus finite geometry may be stated as follows:

(1). There exists at least one line (m-class).

(2). Not all points (elements of S) belong to the same line (m-class).

(3). Not more than one line (m-class) contains any two points (elements of S).

(4). Every line (m-class) contains at least 3 points (elements of S).

(5). No line (m-class) contains more than 3 points (elements of S).

(6). Given a line (m-class) and a point (element of S) not on it, there exists a line (m-class) containing the given point (element of S) which has no point (element of S) in common with the first line (m-class).

(7). Given a line (m-class) and a point (element of S) not on it, there exists not more than one line (m-class) containing the given point (element of S) which has no point (element of S) in common with the first line (m-class).

(8). Given a point (element of S) and a line (m-class) not containing it, there exists a point (element of S) contained in the given line (m-class) which is not on any line (m-class) with the first point (element of S).

(9). Given a point (element of S) and a line (m-class) not containing it, there exists not more than one point (element of S) contained in the given line (m-class) which is not on any line (m-class) with the first point (element of S).

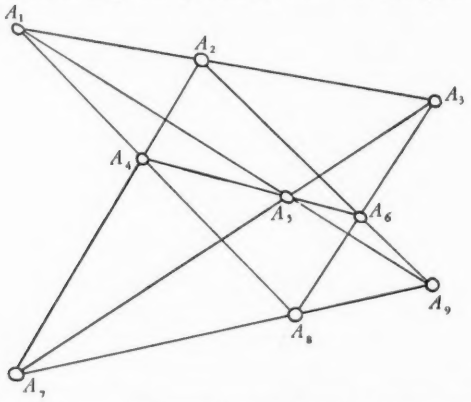
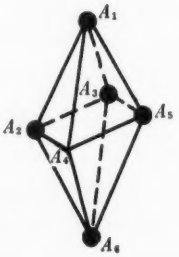


FIG. 3. Geometric diagram for the Pappus finite geometry.

Symbolic diagram where the vertical columns represent lines (m-classes)

$$A_1A_1A_1A_2A_2A_3A_3A_4A_7$$
$$A_2A_4A_5A_4A_6A_5A_6A_5A_8$$
$$A_3A_8A_9A_7A_9A_7A_8A_6A_9$$



The independence of the postulates is shown by the following examples.

(1′). A single point A_1.

(2′). A single line containing points A_1, A_2, A_3.

(3′). The faces of an octahedron, where the faces represent lines.

$(A_1A_2A_3)$, $(A_1A_2A_4)$, $(A_1A_4A_5)$, $(A_1A_3A_5)$,
$(A_6A_2A_3)$, $(A_6A_2A_4)$, $(A_6A_5A_4)$, $(A_6A_3A_5)$.

(4′). A simple quadrilateral (A_1A_2), (A_2A_3), (A_3A_4), (A_4A_1).

(5′). $(A_1A_5A_9\,A_{13})$, $(A_1A_6A_{10}A_{14})$, $(A_1A_7A_{11}A_{15})$, $(A_1A_8A_{12}A_{16})$,
$(A_2A_5A_{10}A_{16})$, $(A_2A_6A_9\,A_{15})$, $(A_2A_7A_{12}A_{14})$, $(A_2A_8A_{11}A_{13})$,
$(A_3A_5A_{11}A_{14})$, $(A_3A_6A_{12}A_{13})$, $(A_3A_7A_9\,A_{16})$, $(A_3A_8A_{10}A_{15})$,
$(A_4A_5A_{12}A_{15})$, $(A_4A_6A_{11}A_{16})$, $(A_4A_7A_{10}A_{13})$, $(A_4A_8A_9\,A_{14})$.

(6′). $(A_1A_2A_3)$, $(A_1A_5A_6)$, $(A_2A_4A_6)$, $(A_3A_4A_5)$.

(7′). $(A_1A_5A_9)$, $(A_1A_6A_{10})$, $(A_1A_7A_{11})$, $(A_1A_8A_{12})$,
$(A_2A_5A_{10})$, $(A_2A_6A_9)$, $(A_2A_7A_{12})$, $(A_2A_8A_{11})$,
$(A_3A_5A_{11})$, $(A_3A_6A_{12})$, $(A_3A_7A_9)$, $(A_3A_8A_{10})$,
$(A_4A_5A_{12})$, $(A_4A_6A_{11})$, $(A_4A_7A_{10})$, $(A_4A_8A_9)$.

(8′). The finite geometry of section 3.

(9′). Two non-intersecting straight lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

The Pappus finite geometry is treated in the book *Fundamentals of Mathematics* by Moses Richardson (Macmillan, 1941).

The duals of the postulates can be proved, showing that the geometry has duality. The geometry has the euclidean property of parallelism of lines. It also has the dual property of parallelism of points.

DEFINITION. *Two points which are not connected by any line will be called parallel points.*

The most important theorems are the following:

THEOREM 1. *If the six points of two (parallel) lines are connected to form a hexagon, the opposite sides intersect in three collinear points.* (Pappus-Pascal theorem).

THEOREM 2. *There are precisely nine points in the geometry.*

THEOREM 3. (Dual of Theorem 2). *There are precisely nine lines in the geometry.*

The Pappus geometry contains no artificial lines and is associated with one of the simplest non-trivial configurations in geometry. The Master's thesis of John E. Darraugh (Brooklyn College, 1940) lists thirty-five theorems for the Pappus finite geometry.

5. The Desargues finite geometry. The postulates of this geometry may be stated as follows:

(1). There exists a point (element of S).

(2). Two distinct points (elements of S) are contained by at most one line (m-class).

DEFINITION. *Line p is called a polar line of point P if no point of p is connected to P by a line. Point P is called a pole of line p if no line through P contains a point of p.*

(3). For every line p (m-class), there is at most one pole P.

(4). There are at least three distinct points (elements of S) on a line (m-class).

(5). For every point (element of S), there exists a polar line (m-class).

(6). If a line p (m-class) does not contain a given point Q (element of S), the polar line q of point Q has a point in common with line p.

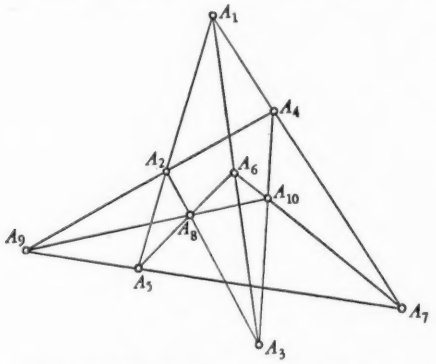


FIG. 4. Geometric diagram for the Desargues finite geometry.

Symbolic diagram where the vertical columns represent lines (m-classes):

$$\begin{array}{l} A_1A_1A_1A_2A_2A_3\ A_5A_5A_6\ A_8 \\ A_2A_3A_4A_3A_4A_4\ A_6A_7A_7\ A_9 \\ A_5A_6A_7A_8A_9A_{10}A_8A_9A_{10}A_{10} \end{array}$$

The independence of the postulates is shown by the following examples.

(1′). The null set—a geometry without points or lines.

(2′). $(A_1A_2A_3)$, $(A_1A_2A_7)$, $(A_1A_3A_8)$, $(A_2A_5A_7)$, $(A_3A_6A_8)$, $(A_4A_5A_6)$, $(A_4A_5A_7)$, $(A_4A_6A_8)$.

(3′). Two lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

(4′). A simple hexagon $A_1A_2A_3A_4A_5A_6$.

$$(A_1A_2),\ (A_2A_3),\ (A_3A_4),\ (A_4A_5),\ (A_5A_6),\ (A_6A_1).$$

(5′). A single point A_1.

(6′). $(A_1A_2\,A_6)$, $(A_1A_3\,A_5)$, $(A_1A_{10}A_{12})$, $(A_2A_3A_4)$, $(A_2A_{10}A_{11})$, $(A_3A_{11}A_{12})$, $(A_4A_5\,A_9)$, $(A_4A_6A_8)$, $(A_5A_6\,A_7)$, $(A_7A_8\,A_{12})$, $(A_7A_9\,A_{11})$, $(A_8A_9A_{10})$.

The Desargues finite geometry has only six postulates, it has duality and polarity, it is non-euclidean in that a line may have as many as three lines

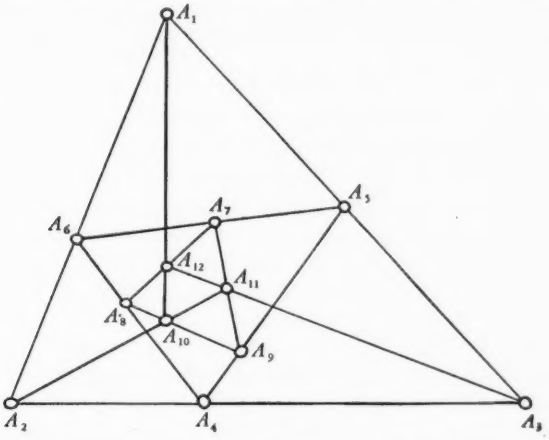


FIG. 5. Four triangles each inscribed in and circumscribed about another triangle of the set.

parallel to it through a given point, and it is associated with a real configuration.

This finite geometry is treated in Part III, Chapter 1 of *Fundamental Mathematics* by Duncan Harkin (Prentice-Hall, 1941).

John E. Darraugh, in his Master's thesis (Brooklyn College, 1940) gives fifty-two theorems for this geometry. Among the most important theorems are the duals of the postulates, and also the following:

THEOREM 1. *If A lies on the polar line of B, then B lies on the polar line of A.*

THEOREM 2. *If b and c are both parallel to a, then b and c intersect in a point.*

THEOREM 3. *There exist precisely ten points.*

THEOREM 4. *There exist precisely ten lines.*

THEOREM 5. *If two triangles are perspective from a point, their corresponding sides intersect in collinear points.* (The Desargues theorem).

ON FREQUENCY DISTRIBUTIONS OF THE QUOTIENT AND OF THE PRODUCT OF TWO STATISTICAL VARIABLES*

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1. Introduction. Given the probability density function (p.f.) for x and y, to find the p.f. for their quotient, and for their product. These problems lie on the surface and in addition to their intrinsic interest they have obvious importance in applications. Surely, even though the mathematical theory of statistics is a relatively new subject, enough study has been given to these two questions to provide the information needed for the more common applications. There is a good deal of literature on the problem for quotients, but nevertheless it is only in some rather special, if important, cases that our knowledge of the distribution of such quantities is satisfactory. But when we are asked, as I have been more than once, by colleagues trying to apply statistical methods to investigations in their own fields, what value can be assigned to the probability that the product of two statistical variables will lie between given values, granting that one knows all about the distributions of the variables separately, even supposing them independent, we simply do not have the information wanted. I do not mean to say that no p.f.'s for products are known; there is at least one important exception; but the state of our ignorance concerning my colleagues's question is embarrassingly complete. My own point of view in considering these two problems is that of a person who would like to see the kind of results which provide satisfactory significance tests for use in applications.

To keep within the space assigned me I shall have to confine my remarks for the most part to three topics chosen from what would amount to a more complete discussion of the subject. First, I shall discuss the situation with regard to the quotient y/x in which x and y obey a normal bivariate p.f. and at the end I shall similarly deal with the product of the same two variables. The second section of the paper will be devoted to quotients of the type of $\alpha_3 = m_3/m_2^{3/2}$, in which m_2 and m_3 are second and third central moments calculated in a sample of N from any infinite universe for which moments of the requisite order exist.

2. Elementary methods for quotients. Two elementary methods for finding the p.f. for a quotient are sufficient for the cases considered here. It is supposed that we are given $f(x, y)$, the joint p.f. for x and y. If x and y are statistically independent, then we are given $f_1(x)$ and $f_2(y)$, and $f(x, y) = f_1(x) \cdot f_2(y)$.

The most obvious thing to do is to set $y = zx$; then the elementary probability for x and y, $f(x, y)\,dx\,dy$, if x and y are continuous variables, is converted to $f(x, zx)\,|x|\,dx\,dz$ (an elementary probability must be positive or zero), the elemen-

* Presented at the Chicago meeting of the Mathematical Association of America on September 1, 1941.

tary probability for x and z. Then the p.f. for z is given by

$$f_3(z) = \int_{\text{range of } x} f(x, zx) \mid x \mid dx,$$

or by

$$f_3(z) = \int_{\text{range of } x} f_1(x) f_2(zx) \mid x \mid dx$$

if x and y are independent.

Professor Huntington has given a derivation of the second form from first principles [1] in case the range for both x and y is the interval $(0, \infty)$. However, the formula is in general valid in the cases in which the sum or integral involved exists.

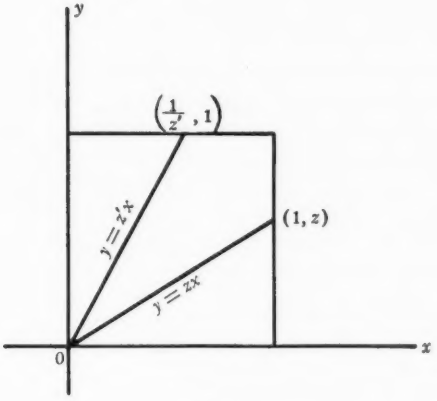

FIG. 1

As a very simple illustration, let x and y be independent and let each obey a rectangular probability law on the interval (0, 1). That is,

$$f_1(x) = f_2(y) = 1, \qquad 0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1,$$
$$= 0, \text{ otherwise.}$$

We have

$$f(z) = \int_0^1 x\, dx = \tfrac{1}{2}, \qquad \text{for} \quad 0 \leqq z \leqq 1,$$
$$= \int_0^{1/z} x\, dx = \frac{1}{2z^2}, \quad \text{for} \quad 1 \leqq z \leqq \infty.$$

(In the case that $z \geqq 1$, $f(zx) \equiv 0$ for $x > 1/z$.)

The second method, only slightly less obvious than the first, is to undertake to find the elementary volume under the frequency surface $p = f(x, y)$ between the two planes $y = zx$ and $y = (z+dz)x$. To do this we try to find the cumulative distribution function (d.f.) for z, $F_3(z)$, which is given by the volume under the frequency surface for which $y \leqq zx$, and then take the derivative (or difference) of $F_3(z)$.

This is also very simply carried out in our hand-picked example. By inspection,

$$F_3(z) = z/2, \qquad 0 \leqq z \leqq 1,$$

$$= 1 - \frac{1}{2z}, \qquad 1 \leqq z \leqq \infty.$$

A more complete discussion of this subject would include an account of the mathematically more sophisticated method of characteristic functions.

3. Quotients of normally distributed variables. Naturally the first important quotient, y/x, studied was that in which x and y are normally distributed. Despite the fact that no fundamental difficulty is encountered, it is nevertheless true that, though there had been earlier papers on the subject, the exact solution in case that x and y obey a normal bivariate p.f. was not published until 1932, by E. C. Fieller [2]. This problem is a bit troublesome and the final solution involves a quadrature, though a tabulated one. Fieller proceeded by finding the d.f. which contains a double quadrature. He remarks, however, that the p.f. could be found more directly by the first method above. If, as we assume,

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\},$$

and if we set

$$\frac{x}{\sigma_x} = u, \quad \frac{y}{\sigma_y} = v, \quad \frac{\mu_x}{\sigma_x} = r_1, \quad \frac{\mu_y}{\sigma_y} = r_2 \quad \frac{v}{u} = z \quad \frac{\sigma_x}{\sigma_y} = w,$$

we have, noting that for a fixed x we must take $dy = |x|\,dz$,

$$f(w) = \frac{\exp\left[-\frac{1}{2(1-\rho^2)}(r_1^2 - 2\rho r_1 r_2 + r_2^2)\right]}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2} + bu\right)|u|\,du,$$

in which,

$$a = \frac{1 - 2\rho w + w^2}{1 - \rho^2} > 0, \qquad \rho^2 < 1,$$

and

$$b = \frac{r_1 - \rho r_2 + (r_2 - \rho r_1)w}{1 - \rho^2}.$$

With r_1 and r_2 not both zero, the final result is,

$$f(w) = \frac{1}{\pi\sqrt{1-\rho^2}} \left\{ \frac{\exp\left[-\frac{1}{2(1-\rho^2)} (r_1^2 - 2\rho r_1 r_2 + r_2^2) \right]}{a} + \frac{b}{a^{3/2}} \exp\left[-\frac{(r_2 - r_1 w)^2}{2a(1-\rho^2)} \right] \int_0^{b/\sqrt{a}} \exp(-t^2/2) dt \right\}.$$

Of course, this can be calculated from existing tables for particular values of w and of the parameters. Tables for $f(z)$ are almost out of the question since the p.f. for z contains not only the parameters r_1, r_2, and ρ but also the ratio σ_x/σ_y. It may be noted that if $r_1 = r_2 = 0$, *i.e.*, if $E(x) = E(y) = 0$, the above result simplifies considerably so that in this case,

$$f(w) = \frac{1}{\pi\sqrt{1-\rho^2}} \cdot \frac{1}{a},$$

or

$$f(z) = \frac{\sqrt{1-\rho^2}\,\sigma_x\sigma_y}{\pi(\sigma_x^2 - 2\rho\sigma_x\sigma_y z + \sigma_x^2 z^2)}.$$

Now the calculation of values of the d.f. is accomplished merely by use of a set of tables of anti-tangents.

But it is to be noted that if x and y obey a normal bivariate p.f. the mean value of y/x does not exist. If we look back at the $f_3(z)$ obtained in the example with rectangular distributions, we see that in that case, too, $E(z)$ does not exist. One will not jump at the conclusion that in general y/x does not possess a mean value. Rather one will have anticipated that allowing zero to be included in the range for x would have some kind of an uncomfortable effect on the distribution of y/x. Of course the situation is simply that if x is distributed normally, or if it follows a rectangular distribution with zero in its range, $E(1/x)$ does not exist. On the other hand, if x has the p.f.

$$\frac{x^{p-1}e^{-x}}{\Gamma(p)}, \qquad p > 0, \qquad 0 \leqq x \leqq \infty,$$

and if ν is the largest positive integer which is $<p$, the first ν moments of $1/x$ are finite. Thus such ratios as Fisher's t and Snedecor's F do in general possess a finite number of moments.

But if $E(y/x)$ does not exist, not even the Tchebychef inequality which states that if x is any statistical variable, the probability that

$$| x - E(x) | \geqq \lambda\sigma_x$$

is $\leqq 1/\lambda^2$, is available to give us any information about the probability that a value of y/x at random will exceed a given value. The much used phrase, "deviation from expected in standard units," is in this case doubly meaningless.

However, there are cases in which $f_1(0) \neq 0$, in which the range of x can be curtailed so as to exclude $x=0$ without doing much violence to the d.f. for y/x. As an example, in the case in which x and y obey a normal bivariate p.f., Geary [3] concluded that if $r_1=\mu_x/\sigma_x$ is large, so that $F_1(0)$ is very small, the quantity

$$\frac{\mu_x z - \mu_y}{(\sigma_x^2 - 2\rho\sigma_x\sigma_y z + \sigma_x^2 z^2)^{1/2}}$$

is approximately normally distributed with zero mean and unit variance.

4. Moment characteristics of quotients. Sometimes the p.f. for a y/x of considerable importance cannot be found because the joint p.f. for x and y is quite unknown. It has been a standard procedure in statistics in case we do not know the p.f. for a variable to try to get the information about it afforded by its moment characteristics, in particular, its mean, its variance, its skewness α_3, and its kurtosis α_4-3 ($\alpha_{k:x}$ is the kth central moment of x divided by σ_x^k). Of course, we ought to have some assurance that the moments sought actually exist, and then we ought to know something about whether or not the actual distribution is such that these first few moment characteristics do enable us to get, at least, first approximations to the probabilities that the variable will fall within a given interval.

A case in point is the ratio $\alpha_3=m_3/m_2^{3/2}$ of problem (2). Even if the population sampled is normally distributed, we cannot produce the joint p.f. for m_2 and m_3. The most that is possible at present is to calculate values of $E(m_2^k m_3^l)$. The computation grows very tedious and involved as $2k+3l$ increases, but the necessary work has been done for $2k+3l \leqq 12$. Can such values be used to obtain approximations to $E(\alpha_2^j)$, at least for small values of j?

Tschuprow [4] used the following method in such a problem. Write

$$E(m_2) = M_2, \qquad E(m_3) = M_3,$$

and then

$$m_2 = M_2 + \delta_2, \qquad m_3 = M_3 + \delta_3$$

with

$$E(\delta_2) = E(\delta_3) = 0.$$

We have then formally, $(j=1)$,

$$E(m_3/m_2^{3/2}) = E\left[\frac{M_3+\delta_3}{(M_2+\delta_2)^{3/2}}\right]$$

$$= \frac{M_3}{M_2^{3/2}}\left(1+\frac{15}{8}\frac{E(\delta_2^2)}{M_2^2}-\frac{35}{16}\frac{E(\delta_2^3)}{M_2^3}+\cdots\right)$$

$$+\frac{1}{M_2^{3/2}}\left(-\frac{3}{2}\frac{E(\delta_2\delta_3)}{M_2}+\frac{15}{8}\frac{E(\delta_2^2\delta_3)}{M_2^2}-\frac{35}{16}\frac{E(\delta_2^3\delta_3)}{M_2^3}+\cdots\right).$$

The values of $E(\delta_2^k\delta_3^l)$ are the ones actually on record, instead of those of $E(m_2^k m_3^l)$, and they have the advantage of being of $O(N^{1-k-l})$. This circumstance seemed to satisfy Tschuprow without further examination that such a series could be used to give a good approximation to the required value for large N's. There is no difficulty in general with the existence of $E(\alpha_3^l)$, in particular, none if we are sampling from a normal universe. But neither of the two series above is in general convergent; they certainly are not if we are sampling from normal.

However, the following argument shows that, nevertheless, we can get good approximations by means of such series for sufficiently large N's. Consider the first of the two series above. Let us restrict ourselves to samples in which $|m_2-M_2| \leqq kM_2$, with $k<1$ and independent of N. Now it is immediately seen that the series is convergent and, moreover, that the error committed by using a given number of terms can be made arbitrarily small with increasing N. For if $\sigma_{m_2}^2$ exists and is $O(1/N)$, as is ordinarily the case, then the probability that $|m_2-M_2| > kM_2$ becomes small with increasing N, and moreover for a fixed h, the difference between $E(\delta_2^h)$ for the curtate range and for the whole range can be made arbitrarily small. Then a given number of terms of the series with moments from the complete range for m_2 can be made to approximate as closely as we please the same number of terms of the series for the curtate range. Thus we can calculate with accuracy the moments of the ratio $m_3/m_2^{3/2}$ with restricted range, since the argument used is readily adapted to each series that arises. But now if the variance of α_3 is finite and $O(1/N)$, as is also usually the case, the moments of α_3 and of α_3 modified as above can also be made to differ as little as we please with increasing N. Therefore the method, so far as the required computations can be carried out, does give good approximations for sufficiently large N's. J. B. D. Derksen has recently discussed these Tschuprow series [5] using some results due to Slutsky [6] in arriving at substantially the argument given above. The practical difficulty arises in estimating how large an N is finally required in each case.

Coming back once more to the case in which x and y obey a normal bivariate p.f., if $|E(x)| \geqq k\sigma_x$ and if we agree to exclude from our distribution values of y/x for which $|x-E(x)| \geqq k\sigma_x$, as I once did [7], Tschuprow's method leads to some

fairly interesting properties of the moment characteristics of such a distribution since now k may be taken to be as large as one chooses.

5. Products; elementary methods. There is at present less to say about the distribution of $w=xy$. Of course there are a few cases in which the p.f. of xy is simple and is simple to obtain. Such a one is that in which $\log x$ and $\log y$ are both normally distributed. An example of a more important and more complicated problem that leads to a relatively simple result is offered by the case in which once more x and y are normally distributed with $E(x)=E(y)=0$. If x and y are independent, we have [8]

$$f(w) = \frac{\sigma_x\sigma_y}{\pi} K_0\left(\frac{w}{\sigma_x\sigma_y}\right),$$

in which $K_0(x)$ is the Bessel function of the second kind with a purely imaginary argument, of zero order. If x and y are correlated, this result is only slightly modified [9].

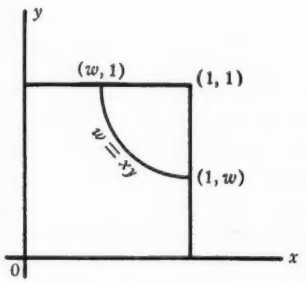

FIG. 2

Of course the same elementary methods which we applied to quotients are also available for use with products. If, again, we suppose that x and y are independent and that each obeys a rectangular probability law on the interval (0, 1), we have

$$f(w) = \int_w^1 \frac{dx}{x} = -\log w, \qquad 0 \leqq w \leqq 1,$$

or

$$F(w) = w + \int_w^1 \frac{w\,dx}{x} = w(1 - \log w).$$

It will be noted that $f(w)$ has an infinite ordinate at the origin, while at that point $F(w)$ is zero and, of course, continuous on the right.

For products there is no question of the existence of moments beyond that of the existence of $E(x^iy^i)$ for the joint p.f. of x and y. That is, the moment char-

acteristics of xy are usually at hand but they do not necessarily afford much information about the probability that $xy \geqq C$.

6. Products of normally distributed variables. I will conclude with sketching briefly the rather incomplete investigation I once made of the case [9], surely important for applications, in which x and y obey a normal bivariate p.f. The simple result given above comes only with the special choice of origin made there. Using the same notation as before, first for x and y independent ($\rho = 0$), it is easy to find the following moment characteristics of xy:

$$E(xy) = \mu_x\mu_y,$$

$$\sigma_{xy}^2 = \sigma_x^2\mu_y^2 + \sigma_y^2\mu_x^2 + \sigma_x^2\sigma_y^2,$$

$$\alpha_{3:xy} = \frac{6r_1r_2}{(r_1^2 + r_2^2 + 1)^{3/2}} \leqq 2\sqrt{3}/3,$$

$$\alpha_{4:xy} - 3 = \frac{6[2(r_1^2 + r_2^2) + 1]}{(r_1^2 + r_2^2 + 1)^2} \leqq 6.$$

Thus in this case the distribution of xy cannot be very asymmetrical, but its kurtosis can be quite excessive.

Here the p.f. can be found as an infinite series. If we set $y = w/x$ and take $dy = dw/|x|$, we obtain, if we put $uv = \zeta$,

$$f(\zeta) = \frac{\exp\left[-(r_1^2 + r_2^2)/2\right]}{2\pi}\left[\psi_1(\zeta) - \psi_2(\zeta)\right],$$

in which

$$\psi_1(\zeta) = \int_0^\infty \exp\left[-(x^2/2 - r_1x - r_2\zeta/x + \zeta^2/x^2)\right]\frac{dx}{x},$$

and $\psi_2(\zeta)$ is the integral of the same function over the interval $(-\infty, 0)$. Now $\psi_1(\zeta)$ and $\psi_2(\zeta)$ can each be found as a convergent series of a not too simple form. Subtracting them, we find,

$$f(\zeta) = \frac{\exp\left[-(r_1^2 + r_2^2)/2\right]}{\pi}\Big[\sum_0 (r_1r_2\zeta)K_0(\zeta) + (r_1^2 + r_2^2)\frac{|\zeta|}{2!}\sum_2 (r_1r_2\zeta)K_1(\zeta)$$
$$+ (r_1^4 + r_2^4)\frac{\zeta^2}{4!}\sum_4 (r_1r_2\zeta)K_2(\zeta) + (r_1^6 + r_2^6)\frac{|\zeta|^3}{6!}\sum_6 (r_1r_2\zeta)K_3(\zeta) + \cdots\Big],$$

in which

$$\sum_k (x) = 1 + \frac{x}{k+1} + \frac{x^2}{(k+2)^2 2!} + \frac{x^3}{(k+3)^3 3!} + \cdots,$$

and is simply related to the Bessel function of the first kind with a purely imagi-

nary argument. Here $K_r(x)$ is the Bessel function of the second kind with a purely imaginary argument.

This series converges for any value of ζ except 0. For $\zeta = 0$, the first term becomes logarithmically infinite at the origin. For $r_1 = r_2 = 0$, we have, as above,

$$f(\zeta) = \frac{K_0(\zeta)}{\pi}.$$

For small values of r_1 and r_2 the series converges rapidly and $f(\zeta)$ can be tabulated and plotted. Unfortunately, for larger values of r_1 and r_2 the series converges more slowly and it is laborious to compute many terms of such a series.

The case in which x and y are not independent ($\rho \neq 0$) turns out not to be essentially more complicated. For this fact and for further details for $\rho = 0$, I refer the reader to my paper cited above [9]. It is evident that though we have a mathematical solution to our problem it falls considerably short of what is desirable for applications.

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PROBLEMS ON MAXIMA AND MINIMA*

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In the simpler problems on maxima and minima studied in the calculus we are led to a function $u(x)$ and we seek values of x for which u has a relative maximum or minimum. If $u(x)$ is differentiable in the interval considered and if the maximum or minimum is assumed at an interior point of the interval, then by Fermat's theorem the derivative du/dx must be zero for the values of x sought. In many cases, however, it is awkward to reduce the problem to the above form and we have instead a function $u(x, y)$ whose maxima and minima we seek to determine under an auxiliary condition, $f(x, y)=0$. Differentiating with respect to x, we have again by Fermat's theorem,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0, \qquad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0.$$

The elimination of dy/dx gives $D(u, f)/D(x, y)=0$ and we solve this equation with $f(x, y)=0$ to find the values of x and y necessary for a maximum or minimum of u.

Instead of this customary analytic approach to the problem, we may employ a partially geometric treatment, as follows. Writing m for dy/dx, we have the following equations for the determination of x, y, m.

$$\frac{\partial u}{\partial x}+m\frac{\partial u}{\partial y}=0, \tag{1}$$

$$\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}=0, \qquad f(x, y)=0. \tag{2}$$

We may interpret (x, y) as the rectangular coördinates of a point in the plane and m as the slope of a line through this point, and thus (x, y, m) form the coördinates of a *linear element*, *i.e.*, a point and a line through it. Equation (1) places one restriction on the three coördinates and thus determines a two parameter family of linear elements, while equations (2) place two restrictions on the coördinates and determine a one parameter family of linear elements. The problem of determining the extrema of u thus resolves itself into that of finding the linear elements common to the two families.

Equation (1) shows that the linear elements of this family lie at each point (x, y) in the direction perpendicular to the gradient $\nabla u \equiv (\partial u/\partial x, \partial u/\partial y)$ and thus in the direction of the curve $u(x, y)=constant$ which passes through this point. This family will also include as singular elements every linear element at the singular points, if any, of the function $u(x, y)$, since the direction of the element becomes indeterminate there. Equations (2) show that the linear elements

* Presented to the Michigan Section of the Mathematical Association of America at Ann Arbor, Michigan, March 9, 1938.

of this family have their points on the curve $f(x, y)=0$ and their lines tangent to this curve at these points. We shall express this by simply saying that the elements are tangent to the curve. This family will also include as singular elements every linear element at the singular points, if any, of the curve $f(x, y)=0$, since the direction of the element becomes indeterminate there. In particular, family (2) might consist merely of the one-parameter family of linear elements at a point; *e.g.*, $f(x, y)=x^2+y^2$.

The problem of finding values of x and y for which extrema of u may occur is thus reduced to finding those points of the plane at which some curve of the family $u(x, y)=constant$ is tangent to the curve $f(x, y)=0$. Geometry is very rich in theorems concerning the constancy of certain point functions $u(x, y)$ along certain plane curves, and by the above analysis each such theorem may yield interesting problems in maxima and minima. Conversely, many problems of the calculus on maxima and minima are almost instantly solved by this analysis and a great deal of light is thrown on all such problems. We present a few examples, starting with a very simple one.

1. If A and B are any two points on a conic, determine the points P on the conic for which the area of the triangle ABP is an extremum.

In this case the curves traces by P when the area u of the triangle ABP remains constant are evidently the straight lines parallel to AB, while for P anywhere on the line through A and B the direction of increase of u is indeterminate.

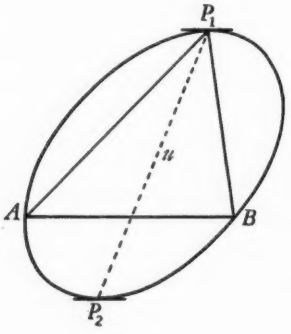

FIG. 1

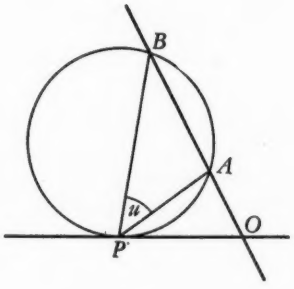

FIG. 2

Family (1) thus consists of all linear elements whose directions are parallel to AB, and includes as singular elements all elements whose points are on AB (extended). Since P must be on the conic, the linear elements of family (2) are those tangent to the conic. The points P_1, P_2, A, B thus constitute the positions of P necessary for an extremum of the area u; P_1 and P_2 being the extremities of the diameter bisecting the chord AB.

The next problem is a generalization of the familiar problem of finding at

what distance one should stand from a tapestry hanging on the wall in order that it subtend the maximum vertical angle at the eye.

2. Two straight roads intersect at any angle at a point O. An enemy column is to traverse one road which is concealed by trees except between two points A and B on the same side of O. At what point P on the other road should a field piece be placed so that the segment AB shall subtend the maximum angle at P?

The curves traced by P when the angle $u=APB$ remains constant are evidently the circular arcs running from A to B, and the regular linear elements of family (1) are those tangent to these arcs. Since P must be a point of the second road, the linear elements of family (2) are tangent to this road. Thus the extrema of u occur for P at the points of tangency of the second road to these circular arcs. But then OP will be the tangent to a circle for which OB and OA are the whole secant and its external segment, and hence $OP^2=OA\cdot OB$. This gives two positions for P, on opposite sides of O and each yielding a maximum of u. The minimum value of u is of course zero, attained when P is at O and yielded by a singular element of family (1), as in Problem 1.

We next consider the familiar "beam around the corner" problem.

3. What is the length of the longest beam that can be moved horizontally from a hall of width a into a hall of width b at right angles to the first?

If we take the outer walls of the halls as coördinate axes, the problem is in effect: What is the length of the shortest line segment with ends on the two axes and passing through the point $A=(a, b)$? Consider any line segment of length u with ends on the two axes. It will, at some point, $P=(x, y)$, be tangent to an

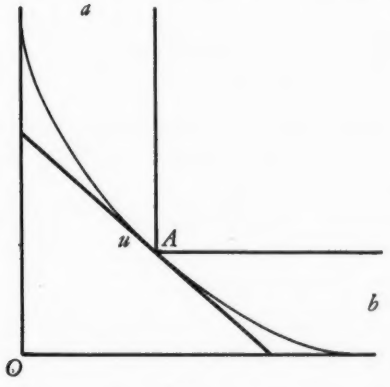

FIG. 3

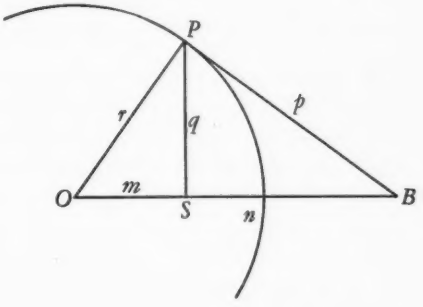

FIG. 4

astroid of the family, $x^{2/3}+y^{2/3}=c^{2/3}$, and it is a familiar property of these astroids that $u^{2/3}=x^{2/3}+y^{2/3}$ and that u consequently remains constant as P traces out the curve. The linear elements of family (1) are thus tangent to the

astroids of this family. The requirement that the line segments pass through A will force P to remain on a certain curve* through A which is the locus of the points P of tangency of lines through A to the astroids of the family, $x^{2/3}+y^{2/3}=c^{2/3}$. The linear elements of family (2) are tangent to this locus. Now let P move along this locus and approach A as a limit point. The limiting position of the line AP is not only the tangent to the locus at A, but also the tangent to the astroid of the family which passes through A. Thus the linear element tangent to this astroid at A belongs to both families and yields the desired minimum value of u. This minimum value is thus given by the equation $u^{2/3}=a^{2/3}+b^{2/3}$.

4. A submarine sights an enemy battleship at a distance and apparently traveling faster than the submarine can go in a direction at an angle to the line of sight. In what direction should the submarine sail to intercept the battleship, if possible?

Let B and S be the initial positions of the battleship and submarine, respectively. Let P be any other point and call $BP=p$, $SP=q$. Evidently the submarine will intercept the battleship, if possible, at the point P of her course at which $u=p/q$ has its maximum value. The curves traced by P when u remains constant are of course the circles having B and S as a pair of inverse points, and the linear elements of family (1) are thus tangent to these circles. Since P must be on the course, supposedly straight, of the battleship, the linear elements of family (2) are tangent to this straight line through B. The position of P for which u is a maximum, being at the common element of these families, must be at the point of tangency of a straight line through B with a circle having B and S as inverse points. Let O be the center of this circle, and call $OS=m$, $OB=n$, $OP=r$. Then since $mn=r^2$, we have $m/r=r/n$ and the triangles OSP and OPB are similar. But then $\angle OSP=\angle OPB=\pi/2$, and the submarine should lay a course at right angles to the bearing of the battleship when sighted.

5. A piece of paper having the form of a right angled triangle of legs a and b has the corner at the right angle folded over to touch the hypotenuse. What is the least possible area of the folded over piece?

Let O and P be the original and final positions of the corner. Then with the notation of the figure we have,

$$OP = \rho = 2m\cos\theta = 2n\sin\theta$$

$$\rho^2 = 4mn\sin\theta\cos\theta = 4u\sin 2\theta,$$

where u is the area of the folded over triangle. The curves traced by P when the area u remains constant are thus the lemniscates $\rho^2=4u\sin 2\theta$ and the linear elements of family (1) are tangent to these lemniscates. Since P must be on the hypotenuse of the triangle, the linear elements of family (2) are tangent to the hypotenuse. For the minimum of u, P must be at the common element of the

* $(x-a)y^{1/3}+(y-b)x^{1/3}=0$.

two families and thus at the point of tangency of the hypotenuse to a member of the family of lemniscates. It is a familiar property of these lemniscates that

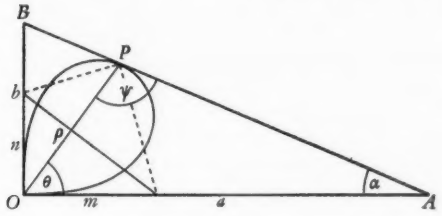

FIG. 5

the angle ψ which the tangent makes with the radius vector ρ is twice the vectorial angle θ. From the triangle OPA we then have,

$$3\theta + \alpha = \pi, \qquad \frac{\rho}{\sin 3\theta} = \frac{a}{\sin 2\theta} \quad \text{where} \quad \tan\alpha = \frac{b}{a}.$$

It follows that $\theta = \frac{1}{3}(\pi - \alpha)$, and the desired minimum value of u is,

$$u = \frac{a^2}{4}\,\frac{\sin^2 3\theta}{\sin^3 2\theta}.$$

In particular, for $\alpha = \pi/2$, the paper becomes rectangular, $\theta = \pi/6$ and $u = \sqrt{6}a^2/18$. For a simple numerical case, rational throughout, take $a = 36/11$, $b = 16/13$. This yields $m = 1$ and $u = 3/8$.

SUMMABILITY AND THE DEFINITION OF A LIMIT

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A classical and well-known method for summation of a sequence s_n is the Hölder method by which

$$s = \lim_{n\to\infty} H_n^{(r)},$$

where

$$H_n^{(r)} = \frac{1}{n+1} \sum_{n_1=0}^{n} \frac{1}{n_1+1} \sum_{n_2=0}^{n_1} \frac{1}{n_2+1} \cdots \sum_{n_r=0}^{n_{r-1}} s_{n_r}. \tag{1}$$

The more widely known Cesàro method most naturally presents itself in the following form

$$s = \lim_{n\to\infty} C_n^{(r)},$$

where

$$C_n^{(r)} = \frac{r!}{(n+1)\cdots(n+r)} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_r=0}^{n_{r-1}} s_{n_r}. \tag{2}$$

It is to be noticed that

$$\frac{(n+1)\cdots(n+r)}{r!} = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_r=0}^{n_{r-1}} 1.$$

Formulas (1) and (2) clearly show a kinship between the Hölder and Cesàro sums. From (2) we write

$$C_n^{(r)} = \frac{r!}{(n+1)\cdots(n+r)} \sum_{n_1=0}^{n} \frac{(n_1+1)\cdots(n_1+r-1)}{(r-1)!} C_{n_1}^{(r-1)} \cdot$$

$$= \frac{n!}{(r+1)\cdots(r+n)} \sum_{n_1=0}^{n} \frac{r(r+1)\cdots(r+n_1-1)}{n_1!} C_{n_1}^{(r-1)}.$$

These are simple expressions for $C_n^{(r)}$ in terms of C_n^{r-1} which together with (1) were used by the author* to prove by mathematical induction that the Hölder and Cesàro methods of summation are equivalent.

Summation by parts according to the formula,

$$\sum_{i=0}^{n} u_i \Delta v_i = u_i v_i \Big]_0^{n+1} - \sum_{i=0}^{n} v_{i+1} \Delta u_i,$$

applied successively to (2) yields

$$C_n^{(r)} = \frac{r!}{(n+1)\cdots(n+r)} \sum_{n_1=0}^{n} \frac{(n-n_1+1)\cdots(n-n_1+r-1)}{(r-1)!} s_{n_1} \tag{3}$$

$$= \frac{n!}{(r+1)\cdots(r+n)} \sum_{n_1=0}^{n} \frac{(r-1)r\cdots(n-n_1+r-1)}{(n-n_1)!(r-1)} s_{n_1}.$$

This last is the form in which the Cesàro definition is usually given. There is an advantage over (2) in that non-integral values of r are permitted.

It is to be observed that the Cesàro method is a special case of summation by triangular matrix,

$$s = \lim_{n\to\infty} \sum_{i=0}^{n} K(n, i) s_i, \tag{4}$$

which has been extensively studied.† It is also to be remarked that definitions of summability by triangular matrix are extensions of the ordinary definitions of the limit of a sequence in case the method of summation is regular.

Now if we consider a function $f(x)$ of the continuous real variable x, similar methods of "summation" of $f(x)$ immediately suggest themselves. For the Hölder definition we have;

* See Bull. Amer. Math. Soc., vol. 33, p. 301.

† See, for example, Fort, Infinite Series, p. 241.

The function $f(x)$ is summable to A if $\lim_{x\to\infty} H^{(r)}(x)=A$ where,

$$(5)\qquad H^{(r)}(x)=\frac{1}{x}\int_0^x\frac{1}{x_1}\int_0^{x_1}\cdots\frac{1}{x_{r-1}}\int_0^{x_{r-1}}f(x_r)dx_1dx_2\cdots dx_r.$$

The definition of integration can be, for example, the Riemann definition.

Likewise the Cesàro definition suggests,

The function $f(x)$ is summable to A if $\lim_{x\to\infty} C_1^{(r)}(x)=A$, where

$$(6)\qquad C_1^{(r)}(x)=\frac{r!}{x^r}\int_0^x\int_0^{x_1}\cdots\int_0^{x_{r-1}}f(x_r)dx_1dx_2\cdots dx_r.$$

If integration by parts is applied to formula (6) we have

$$C_1^{(r)}(x)=\frac{r!}{x^r}\int_0^x\frac{(x-\alpha)^{r-1}}{(r-1)!}f(\alpha)d\alpha=\int_0^x\frac{r(x-\alpha)^{r-1}}{x^r}f(\alpha)d\alpha.$$

It is to be observed that this integral is a special case of

$$(7)\qquad \int_0^x K(x,\alpha)f(\alpha)d\alpha.$$

Integration by parts will also reduce (5) to the form (7). Form (7) can always be written

$$(8)\qquad \psi(x)=\int_0^\infty K(x,\alpha)f(\alpha)d\alpha$$

by simply defining $K(x,\alpha)$ as zero when $x>\alpha$. We shall call this the transformation by integration with kernel K and $\lim_{x\to\infty}\psi(x)$ the limit of $f(x)$ by integration with kernel K. The integral (8) is a form which has been much studied.*

It is proposed to set up an analogous form to (7) from the theory of finite differences. Note that $\int_0^x f(x)dx$ is a continuous function which is a solution of the differential equation $dy/dx=f(x)$. We propose to utilize the solution of the difference equation, $\Delta_\omega y=f(x)$ called by Nörlund† a "sum" and denoted by him by $S_c^x f(t)\Delta_\omega t$. The results obtained are far from complete judged by results long known for summation by triangular matrix. They do not seem to be without interest, however, and may be suggestive for further research. The definition of Nörlund "sum" adopted throughout the present paper is

$$\overset{x}{\underset{c}{S}}f(t)\Delta_\omega t=\lim_{\mu\to 0}\left[\int_c^\infty e^{-\mu t}f(t)dt-\omega\sum_{i=0}^{\infty}e^{-\mu(x+i\omega)}f(x+i\omega)\right],$$

where $\mu\geqq 0$, $\omega>0$. Wherever the symbol or its equivalent is used existence is assumed. The integral of the definition is assumed to be an ordinary improper

* See for example R. P. Agnew, Bull., Amer. Math. Soc., vol. 45, p. 689.

† See N. E. Nörlund, Differenzenrechnung, p. 43.

integral, where the Riemann definition of integration is used for the finite interval. This "sum" is a continuous function of the continuous variable x, having many resemblances to a definite integral.

Consider

$$\phi(x) = \overset{x}{\underset{0}{S}} K(x, t)f(t)\Delta_\omega t. \tag{9}$$

Throughout this paper we shall assume all variables real. We call (9) the transformation by continuous summation with kernel K and $\lim_{x\to\infty} \phi(x)$ the limit by continuous summation with kernel K. However, whenever the word, limit, is used alone it will refer to the limit in the ordinary (ϵ, δ) sense. For purposes of illustration we define by analogy with (3) the Cesàro limit of order r of $f(x)$ by this definition as $\lim_{x\to\infty} C_2^{(r)}(x)$ where

$$C_2^{(r)}(x) = \frac{r\Gamma\left(\frac{x}{\omega}+1\right)}{\Gamma\left(\frac{x}{\omega}+r+1\right)} \overset{x+\omega}{\underset{0}{S}} \frac{\Gamma\left(\frac{x-t}{\omega}+r\right)}{\Gamma\left(\frac{x-t}{\omega}+1\right)} f(t)\Delta_\omega t.$$

Analogous forms suggest themselves such as $\lim_{x\to\infty} C_3^r(x)$ where

$$C_3^{(r)}(x) = r\frac{Q_r(x)}{B_r^{(r)}(x)}$$

with

$$Q_r(x) = \overset{x+\omega}{\underset{0}{S}} B_{r-1}^{(r-1)}(t)f(t)\Delta_\omega t$$

where $B_j^{(j)}(x)$ is the Bernoulli polynomial* of order j and degree j with difference intervals, $\omega_1, \omega_2, \cdots, \omega_j$. Forms analogous to the Hölder sum could also be suggested.

At this point we introduce the following definitions.

The transformation (9) *is said to be limit-producing for the function* $f(x)$ *if* $\lim_{x\to\infty} \phi(x)$ *exists. It is said to be regular if, whenever* $\lim_{x\to\infty} f(x) = A$, $\lim_{x\to\infty} \phi(x) = A$ *for those functions for which* $\phi(x)$ *exists*† *for all positive values of* x.

THEOREM I. *Necessary conditions that transformation* (9) *be regular are*

$$\lim_{x\to\infty} \overset{x}{\underset{0}{S}} K(x, t)\Delta_\omega t = 1, \tag{a}$$

$$\lim_{x\to\infty} \int_{t_0}^{t_0+\delta} K(x, t)dt = 0, \tag{b}$$

* N. E. Nörlund, *loc. cit.*, p. 129.

† See, for example, Nörlund, *loc. cit.*, p. 47.

whenever $t_0 \geqq 0$ and $\delta \geqq 0$ provided that the sum and integral exist.

Consider the function $f(t) \equiv 1$. Condition (a) results. Consider the function, $f(t) = 1$ when $t_0 \leqq t \leqq t_0 + \delta$, $f(t) = 0$ when $t > t_0 + \delta$ or $t < t_0$; condition (b) results.

Next suppose $K(x, t) = 0$ if $t > x$, then*

$$\phi(x) = \int_0^x K(x, t)f(t)dt - \omega K(x, x)f(x). \tag{10}$$

Let $f(x) \equiv 1$ and $K(x, x) \equiv 1$, $K(x, t) = 1/(t+1)^2$, $t \neq x$; then the limit by integration with kernel K is 1 but the limit by continuous summation is $1 - \omega$. In fact, from formula (10) we readily conclude the following theorem.

THEOREM II. *Functions K and f can be defined such that the limit by integration with kernel K is different from the limit by continuous summation with the same kernel K. In fact one limit may exist without the existence of the other.*

Likewise in case the expressions written converge†

$$\phi(x) = \int_0^\infty K(x, t)f(t)dt - \omega \sum_{i=0}^{\infty} K(x, x + i\omega)f(x + i\omega).$$

From this we have the following theorem.

THEOREM III. *A sufficient condition that the transformation by summation with kernel K be regular is that the transformation by integration with kernel K be regular and that*

$$\lim_{x \to \infty} \sum_{i=0}^{\infty} K(x, x + i\omega)f(x + i\omega) = 0.$$

From our definition we write

$$\phi(x) = \lim_{\mu \to 0+} \left[\sum_{i=0}^{\infty} \left(\int_{\delta + i\omega}^{\delta + (i+1)\omega} e^{-\mu t} K(x, t)f(t)dt - \omega e^{-\mu(\delta + i\omega)} K(x, \delta + i\omega)f(\delta + i\omega) \right) \right.$$
$$\left. + \omega \sum_{i=0}^{(x-\delta-\omega)/\omega} e^{-\mu(\delta + i\omega)} K(x, \delta + i\omega)f(\delta + i\omega) + \int_0^{\delta} e^{-\mu t} K(x, t)f(t)dt \right].$$

where $0 \leqq \delta < \omega$ and $(x - \delta) \equiv 0 \pmod{\omega}$. Now in case $tK(x, t)f(t)$ is positive monotonic decreasing in t for each x, each terms of the series

$$\sum_{i=0}^{\infty} \left| \int_{\delta + i\omega}^{\delta + (i+1)\omega} e^{-\mu t} K(x, t)f(t)dt - \omega e^{-\mu(\delta + i\omega)} K(x, \delta + i\omega)f(\delta + i\omega) \right| \tag{11}$$

increases when μ decreases. Moreover, (12) converges when $\mu = 0$ as is shown

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* See L. M. Milne-Thomson, Calculus of Finite Differences, p. 201.

† See L. M. Milne-Thomson, *loc. cit.*, p. 201.

by comparison with the telescopic series

$$\sum_{i=0}^{\infty} (K(x, \delta + i\omega)f(\delta + i\omega) - K(x, \delta + (i+1)\omega)f(\delta + (i+1)\omega)).$$

Hence (11) converges uniformly in μ by the Weierstrass test* using for comparison series (11) itself when $\mu = 0$. Under these conditions

$$\phi(x) = \sum_{i=0}^{\infty} \left(\int_{\delta+i\omega}^{\delta+(i+1)\omega} K(x, t)f(t)dt - \omega K(x, \delta + i\omega)f(\delta + i\omega) \right) + \omega \sum_{i=0}^{(x-\delta-\omega)/\omega} K(x, \delta + i\omega)f(\delta + i\omega) + \int_0^{\delta} K(x, t)f(t)dt, \tag{12}$$

where $(x-\delta) \equiv 0 \pmod{\omega}$ and $0 \leqq \delta < \omega$.

From formula (12) we draw some conclusions as expressed in the following theorems provided, of course, formula (12) holds.

THEOREM IV. *The following conditions together are sufficient that transformation* (9) *be limit producing*:

(a) $tK(x, t)f(t)$ *be positive monotonic decreasing in* t *for each* x;

(b) $$\sum_{i=0}^{(x-\delta-\omega)/\omega} K(x, \delta + i\omega)f(\delta + i\omega)$$

approach one and the same limit as $x \to \infty$ *through any set of values such that* $(x-\delta) \equiv 0 \pmod{\omega}$, *where* $0 \leqq \delta < \omega$;

(c) $$\sum_{i=0}^{\infty} \left(\int_{\delta+i\omega}^{\delta+(i+1)\omega} K(x, t)f(t)dt - \omega K(x, \delta + i\omega)f(\delta + i\omega) \right)$$

approach one and the same limit as $x \to \infty$ *through any set of values such that* $(x-\delta) \equiv 0 \pmod{\omega}$ *and* $0 \leqq \delta < \omega$;

(d) $$\lim_{x \to \infty} \int_0^{\delta} K(x, t)f(t)dt = 0, \qquad 0 \leqq \delta < \omega.$$

From Theorem IV we conclude the following theorem by showing that the conditions stated in it are sufficient for the conditions of Theorem IV. Details will be omitted.

THEOREM V. *The following taken together are sufficient conditions that transformation* (9) *be limit-producing*:

(a) $tK(x, t)f(t)$ *be positive monotonic decreasing in* t *for each* x;

(b) $\lim_{x \to \infty} K(x, t) = 0$ *uniformly in each interval,* $0 \leqq t \leqq \delta$, $\delta + i\omega \leqq t \leqq \delta + (i+1)\omega$;

* See, for example, Fort, Infinite Series, p. 101.

(c) *$d/dtK(x, t)f(t)$ be monotonic increasing in t for each x.*

(d) *The transformation*

$$\sum_{i=0}^{(x-\omega)/\omega} K(x-\omega, i\omega)f(i\omega)$$

be limit-producing when $x \equiv 0 \pmod{\omega}$.

Now by the Euler-Maclaurin summation formula in case certain conditions on $K(x, t)f(t)$ are fulfilled* we can write

$$(13)\quad \sum_{0}^{x+\omega} SK(x, t)f(t)\Delta_\omega t = \int_0^x K(x, t)f(t)dt + \sum_{v=1}^{m} \frac{\omega^v}{v!} B_v \left[\frac{\partial^{v-1}}{\partial\alpha^{v-1}} K(x, \alpha)f(\alpha) \right]_{\alpha=x} + \frac{\omega^{m+1}}{m!} \int_0^\infty P_m(1-t) \frac{\partial^m}{\partial t^m} [K(x, x+\omega t)f(x+\omega t)]dt.$$

From this we conclude immediately the following theorem.

THEOREM VI. *If, in addition to* (13),

$$\int_0^\infty \left| \frac{\partial^m}{\partial t^m} K(x, x+\omega t)f(x+\omega t) \right| dt$$

converges uniformly in x then

$$\lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \left(\int_0^x K(x, t)f(t)dt + \sum_{v=1}^{m} \frac{\omega^v}{v!} B_v \left[\frac{\partial^{v-1}}{\partial\alpha^{v-1}} K(x, \alpha)f(\alpha) \right]_{\alpha=x} \right).$$

From Theorem VI we conclude the following theorems.

THEOREM VII. *Sufficient conditions, in addition to* (13), *that the transformation* (9) *be limit-producing are the following taken together*;

(a) *The integral transformation* (7) *with kernel K be limit-producing*;

(b) $$\frac{\partial^n}{\partial\alpha^n} K(x, \alpha) \Big]_{\alpha=x}$$

exist and approach a limit when $x\to\infty$, $n=1, 2, \cdots, m$;

(c) $$\int_0^\infty \left| \frac{\partial^n}{\partial t^n} K(x, x+\omega t) \right| dt,$$

$n=1, 2, \cdots, m$, converge uniformly in x;

(d) *$f(x), f'(x), \cdots, f^{(m)}(x)$ approach limits when $x\to\infty$.*

* See L. M. Milne-Thomson, *loc. cit.*, p. 212.

THEOREM VIII. *Sufficient conditions, in addition to* (13), *that* $\phi(x) \to A$ *are the following taken together*:

(a) $f(x) \to A$ *when* $x \to \infty$ *and* $f^{(m)}(x) \to 0$ *when* $m \geqq 1$;

(b) $$\frac{\partial^n}{\partial \alpha^n} K(x, \alpha) \Big]_{\alpha = x}$$

approach a limit, when $x \to \infty$, $n = 0, 1, \cdots, m$;

(c) $$\int_0^\infty \left| \frac{\partial^n}{\partial t^n} K(x, x + \omega t) \right| dt$$

$n = 1, \cdots, m$, *converge uniformly in* x;

(d) *the transformation* (7) *be regular.*

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A NEW SYSTEM FOR PLAYING THE GAME OF NIM

D. P. MCINTYRE, Victoria, B. C.

The game of Nim is played by two players. Matches or counters are arranged in any number of groups with any number in each. The players alternately draw one or more counters from any one group and the player who draws last wins.

The theory of this game making use of the binary representation of numbers has already been worked out* and shows that the first player, provided he knows the system, can, with few exceptions, win. The rare exceptions occur when the initial arrangement is favorable to his opponent, but in any case a player who knows the system can usually defeat one who does not.

The theory presented here is based on the quaternary representation of numbers, which is somewhat simpler of application than the binary representation since it involves fewer terms.

1. *Definition of the quaternary representation and its coefficients.* Let the number of groups in the game be n, and the number of elements in the ith group be x_i, where $1 \leqq i \leqq n$. Then the numbers x_i can be expressed in the form

$$x_i = \sum_{j=0}^{\infty} a_{ij} 4^j, \qquad 0 \leqq a_{ij} \leqq 3, \qquad 1 \leqq i \leqq n.$$

* See, for example, C. L. Bouton, Annals of Math., ser. II, vol. 3, 1901, p. 35.

2. *Definition of balanced systems of groups.* The set of coefficients a_{ij} for a fixed j is said to be balanced if it can be broken up into one or more sets each of which is either a pair (*e.g.*, 2, 2), the set 1, 2, 3, or the single element 0. For example, the set 1, 3, 0, 1, 1, 2, 0, 3, 3 is balanced since it consists of the pairs 1, 1 and 3, 3 and the set 1, 2, 3 along with two 0's. The system of groups is known as a balanced system if the coefficients a_{ij} are balanced for every fixed value of j. For example, the following system is balanced:

$$6 = 1\cdot 4 + 2,$$
$$5 = 1\cdot 4 + 1,$$
$$3 = 0\cdot 4 + 3.$$

3. *Any unbalanced system can be balanced by the removal of one or more units from one particular group.* We will consider only games of more than two groups since the latter systems can obviously be balanced by evening up the groups. If any of the sets of coefficients a_{ij} for fixed j are unbalanced, the method of cancelling out pairs and sets of 1, 2, 3 will reduce the set to at most two digits. Thus any single set a_{ij} for fixed j can be balanced by subtracting or adding enough from one of the digits to balance the other or by adding enough to a zero digit to make a balanced set or by subtracting the odd digit. The set may also be balanced by altering one and only one previously chosen coefficient. For by keeping this coefficient fixed and cancelling out zero terms, the set can be reduced to at most three terms. If there were more than three terms, there would be pairs or sets of 1, 2, 3 among the unfixed terms. The set can then be balanced by altering the chosen coefficient to complete the other one or two terms if any exist.

Let k be the highest value of j for which the coefficients a_{ij} are unbalanced. Reasoning as before, these can be balanced by the removal of a small multiple of 4^k from one of the groups of x_i units. Denote this group by $i=m$. Thus we have available at least 4^k units to balance the remaining coefficients a_{ij} for each fixed j by altering the coefficients a_{mj}, $(j \leqq k-1)$. The most that we will be required to add to balance any set will be $3\cdot 4^j$ units. Thus the maximum number of units required for the balancing of the remaining k sets is

$$\sum_{j=0}^{k-1} 3\cdot 4^j = 4^k - 1 \text{ units.}$$

But we have available at least 4^k units. Hence it is always possible to balance any given unbalanced set of groups by subtracting one or more units from one of the groups.

4. *A balanced system is a losing game for the first player.* Since all the groups are diminishing in size during the play they must all eventually reduce to digits of less than 4. If a player has been presented with a balanced system at each

turn he will eventually have to play against a system of groups composed of digits none of which is greater than three. Thus the system contains only pairs and sets of 1, 2, 3 which is certainly a loss.

5. *Quick setting up of balanced groups.* In order to be sure of winning it is necessary to present an opponent with a balanced system with every move. The easiest way to do this is to break the numbers down into quaternary form and to decide which one has the largest unbalanced term. This group is the one which must be cut down. Ignore this group and consider only the others of the system, and by cancelling coefficients determine the number which will balance the system. With terms of order higher than 4^0 it is easier to handle the whole term rather than the coefficient; *i.e.*, consider the terms to be 0, 4, 8, 12 or 0, 16, 32, 48 instead of 0, 1, 2, 3.

Example. Balance the system of groups 8, 36, 17, 31, 14.

The group 36 contains the term $2 \cdot 4^2$ or 32 which cannot be balanced by any of the other groups; hence this term must be cut down. Considering only the other groups, we have

Order	8	36	17	31	14	Group required to complete system
4^2	0		~~16~~	~~16~~	0	0
4^1	8		0	~~12~~	~~12~~	8
4^0	0		~~1~~	~~3~~	~~2~~	0
						8

Thus if the group 36 is reduced to 8, the system will be balanced.

If the opponent confronted with a balanced system does not reduce a group below the highest multiple of 4 contained in the group, he can only succeed in deranging the coefficients of 4^0 and the system can again be balanced by considering only these coefficients. For instance, in the example above, if the opponent playing on the system 8, 8, 17, 31, 14 which is balanced decides to reduce the group of 31 by 1, 2, or 3 (*i.e.*, he does not reduce the 31 to a number less than 28), he only succeeds in deranging the coefficients of 4^0 and the first player may again balance the system by removing a sufficient number of units of the same order from the 17 or 14 groups.

A SECOND NOTE ON AUTOPOLAR CURVES

MALCOLM FOSTER, Wesleyan University

1. Introduction. In a recent paper* the author has considered those curves which are autopolar with respect to a parabola. Such curves are here regarded as special solutions of those differential equations which are invariant under the Legendrian dual transformation for which this parabola is the conic of reference. It is the purpose of this brief note to state certain theorems which follow quite readily from my first paper on the subject.

Relative to a given conic of reference, two curves are said to be polar reciprocals if the polars of the points of either curve envelope the other; an autopolar curve is its own polar reciprocal. We shall always use the parabola $2\eta = \xi^2$ as the conic of reference.

2. The conjugate circle. Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on an autopolar curve C are called a conjugate pair if the tangent to C at either is the polar of the other with respect to the given parabola. Associated with this pair, let us consider the points Q and R in which the pairs of tangents and normals at P_1 and P_2 intersect. The four points P_1, P_2, Q, and R evidently lie on a circle for which QR is a diameter; this circle we shall call a "conjugate circle." The coördinates of its center are†

$$\left(\frac{y_1 - y_2 + y_1^2 - y_2^2 + x_1^2 - x_2^2}{2}, -\frac{1}{2}\right).$$

Hence we have the following:

THEOREM 1. *For any curve autopolar with respect to the parabola $2\eta = \xi^2$, the locus of the centers of the conjugate circles is the directrix of the conic of reference.*

3. Two metric properties. Let P_1' and P_2' be the points on the parabola of reference whose abscissas are respectively equal to the abscissas of a conjugate pair, P_1 and P_2. The product of the curvatures of the parabola at P_1' and P_2' is readily found to be

$$K_p = \frac{1}{[(1 + x_1^2)(1 + x_2^2)]^{3/2}};$$

and the product of the curvatures of the autopolar curve at P_1 and P_2 is

$$K_a = \frac{y_1'' y_2''}{[(1 + x_1^2)(1 + x_2^2)]^{3/2}}.$$

But for this dual transformation,‡ $y_1'' y_2'' = 1$. We therefore have the following:

* Malcolm Foster, Note on autopolar curves, Bulletin of the American Mathematical Society' vol. 47, 1941, pp. 247–253.

† Foster, *loc. cit.* See equations (14) and (15), p. 250.

‡ Foster, *loc. cit.*, equation (2), p. 247.

THEOREM 2. *For any curve autopolar with respect to the parabola* $2\eta = \xi^2$, *the product of the curvatures at a conjugate pair is equal to the product of the curvatures of the parabola at those points whose abscissas are equal to the abscissas of the conjugate pair.*

Since in the above demonstration we have made no use of the condition of autopolarity,* it follows that Theorem 2 is also valid in the case of corresponding points on pairs of polar reciprocal curves.

We readily find that the equation of the tangent to the parabola of reference at P_1' is $2x_1x - 2y - x_1^2 = 0$, and that the coördinates of the midpoint S of the segment which joins P_1 and P_2 are

$$x = (x_1 + x_2)/2, \qquad y = (y_1 + y_2)/2 = x_1x_2/2.$$

These coördinates, however, satisfy the above equation; hence, since it is evident that the coördinates of S must also satisfy the equation of the tangent to the parabola at P_2', we have the following:

THEOREM 3. *For any curve autopolar with respect to the parabola* $2\eta = \xi^2$, *the line which joins any conjugate pair* P_1 *and* P_2 *is concurrent with the tangents to the parabola of reference at* P_1' *and* P_2' *at* S, *the midpoint of the segment* P_1P_2.

Since Q is the pole of P_1P_2, it follows that the polar of S must pass through Q. Hence Q, P_1', and P_2' are collinear.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Development of Rationalism and Empiricism. By G. de Sentillana and E. Zilsel (International Encyclopedia of Unified Science, Volume II, Number 8.) Chicago, University of Chicago Press, 1941. 8+94 pages. $1.00.

The Principles of Financial and Statistical Mathematics. Revised edition. By Maximilian Philip. New York, Prentice-Hall, Inc., 1941. 16+335 pages. $3.50.

A Manual of Problems in Statistics. By S. Dayton. New York, Henry Holt and Company, 1941. 163 pages. $0.90.

Tools. A Mathematical Sketch and Model Book. By R. C. Yates. Baton Rouge, Louisiana State University Press, 1941. 194 pages. $1.60.

Factor Analysis; A Synthesis of Factorial Methods. By K. J. Holzinger and

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* Foster, *loc. cit.*, equation (10), p. 249.

H. H. Harman. Illinois, The University of Chicago Press, 1941. 12+417 pages. $5.00.

What is Mathematics? By R. Courant and H. Robbins. London, New York and Toronto, Oxford University Press, 1941. 19+521 pages.

Fourier Series and Orthogonal Polynomials. By D. Jackson. The Carus Monographs, No. 6. Oberlin, Ohio. The Mathematical Association of America, 1941. 12+234 pages. $2.00; to members, $1.25.

Early Military Books in the University of Michigan Libraries. By T. M. Spaulding and L. C. Karpinski. Ann Arbor, Michigan, The University of Michigan Press, 1941. 16+45 pages; 25 plates.

Lectures in Topology. Edited by R. L. Wilder and W. L. Ayres. The University of Michigan Conference of 1941. Ann Arbor, Michigan, The University of Michigan Press; London, Humphrey Milford and Oxford University Press, 1941. 7+316 pages. $3.00.

REVIEWS

Fundamental Theorems of Orthographic Axonometry and their Value in Picturization. By W. H. Roever. Washington University Studies, New Series, Science and Technology, No. 12. St. Louis, Missouri, Washington University Press, 1941. 47 pages. $1.00.

This little pamphlet gives a vivid picture of the meaning of axonometry by means of a detailed comparison of the representation of a bracket-shaped figure by the Mongean method and by the axonometric method. Then the fundamental theorems of the latter are derived. The foreshortening ratios, axonomatic scales, and the necessary analytic relations are carefully explained and derived, followed by a detailed analysis of the theorems of Schwarz and of Gauss. These are applied to the solutions of two fundamental problems:

I. Given the axonometric projections of the coördinate axes, to find the foreshortening ratios.

II. Given the foreshortening ratios, to find the mutual inclinations of the projections of the coördinate axes.

The style is simple and direct, and the 24 figures are excellent.

VIRGIL SNYDER

Odd Numbers or Arithmetic Revisited. By Herbert McKay. New York, The Macmillan Company, 1940. 1+215 pages. $2.50.

This book is an interesting presentation of arithmetic for the average layman. It is by no means a text-book. Instead, it is written in a so-called popular manner so as to make it especially appealing to those having little knowledge of arithmetic and to arouse in them an appreciation not only of its usefulness and application but also of arithmetic itself. The author points out that "Arithmetic is usually regarded as the Cinderella of Mathematics, the drudge whose duty it is to do everything that is dull." He deplores the opinion that mathematics is

dull, and opines that the understanding and manipulation of numbers is exciting. He endeavors to substantiate this fact and in the reviewer's opinion does it fairly well.

The material covered includes powers, logarithms, proportion, weights and measures, the arithmetic mean, comparisons, approximations, multiplication and division, tables, units, oddities of numbers, and scales of notation. He tries to make large numbers understandable. Numerous practical examples and illustrations assist to clarify the topics discussed and add to the popular interest.

The author mentions the error which caused the Italians to celebrate the bimillenary of Augustus a year too soon, and H. G. Wells's error in allowing his man in the moon to do some fantastic jumping. A simple explanation is given of the basis of logarithms. Arguments are presented both for and against the decimal and duodecimal systems. He explains a number of tricks in reckoning. In connection with weights and measures, the author defines a unit as "a convenient amount of the quantity to be measured." An explanation is given of how the different units came into being.

The book is enjoyable, stimulating, and amusing.

F. M. WEIDA

The Elements of Statistics. By E. B. Mode. New York, Prentice-Hall, Inc., 1941. 16+378 pages. $3.50.

In the announcement of a recently published work on Statistical Bibliography, the author states that the first objective of his book is: "To make students and teachers of statistics more keenly aware of the inadequacy of much of what is now presented in textbooks and classes despite the fact that such statistical techniques are incorrect, inefficient, and obsolete." By stressing these shortcomings, he expresses exactly what so many research workers and teachers in statistics are feeling today.

All the more do we welcome a book like that by E. B. Mode which, elementary as it is, gives a clear and correct presentation of the problems it deals with. These problems are manifold. Besides the basic concepts of descriptive statistics, such as frequency distributions, averages, moments of one-dimensional and bivariate distributions, we find a chapter on index numbers and some remarks on time series. These sections form the main part of the book, whereas the references to theory of probability and to problems of small samples are rather brief. The use of various sorts of graph paper is as welcome as the valuable introduction to approximate computation. The exercises accompanying the different chapters, "none of which have been borrowed from other textbooks," are interesting and stimulating. The proofs of simple mathematical theorems are generally given; in a few other cases, the reader gets a correct statement of the theorem. The book will certainly be very useful for all statisticians not primarily interested in theoretical statistics.

HILDA P. GEIRINGER

Algebra. A Text-book of Determinants, Matrices, and Algebraic Forms. By W. L. Ferrar. Oxford, Clarendon Press, 1941. vii+202 pages. $3.50.

The purpose of this book is to provide material for an undergraduate or a first year graduate course in the elements of the theories of matrices and algebraic forms. The book is in three divisions: I *Determinants*, II *Matrices*, III *Linear and Quadratic Forms*. In a brief treatment, as this volume is, a careful selection of material is necessary. On the whole, the author's choice of subject matter is judicious, particularly in the chapter on invariants in Division III. The omission from Division II of the topics of similarity and matrices with polynomial elements may be questionable.

For the most part, the clarity of the presentation is commendable and the author in preparing the book has kept the student uppermost in his mind. However, the statement of the separate corollaries to Theorem 7, p. 16, would be an insult to the intelligence of even a dull student. Following are listed several features which appealed to the reviewer: a novel definition of a determinant is used in Division I (which later affords a very neat proof of the theorem about the evenness or oddness of a permutation); an elegant proof of the theorem of Frobenius concerning the characteristic roots of a rational function of a matrix; a clear and elementary proof of the theorem of Darboux, that the principal minors of the matrix of a positive definite quadratic form are all positive; the well chosen problems (which do not include many arithmetical examples, however) particularly those in connection with form theory, invariant theory, and manipulation of determinants.

There are a number of features which most mathematicians will find objectionable. Those of a general occurrence are: sporadic use of dummy summation indices; lack of distinction between a square array and a determinant; use of $\times$ for matric multiplication; use of *ordinary* as a synonym for *non-singular*; use of *transformation of* instead of *transform of*; failure to employ matric theory to advantage in the development of form theory. Also notable are the following: the *first step* of p. 9 is superfluous; the laws of exponents for non-singular matrices are awkwardly handled; the lemma on p. 165 could be made much more general without altering the proof; the matrix J_1 on p. 165 is its own inverse, and noting that fact would make the presentation clearer. The language in several places is unjustifiably vague (*cf.* pp. 17, 30, 189, 157 [Ex. 2]).

More serious than the foregoing, however, is the excessive number of errors and slights of rigor which mar the book: *cf.* the definition of field, pp. 2 and 119; p. 35, Ex. 4; p. 107, Ex. 9; p. 60, proof of Theorem 21; p. 115, neglecting to prove that the inverse of an elementary transformation is an elementary transformation; p. 156, vitiation of proof of Theorem 48, due to failure to prove existence of a λ_1; p. 154, equation (4) is not justified by the preceding argument; p. 156, Ex. 1 as stated is false.

The format of the book is pleasing and the typography is excellent (the reviewer noted only three typographical errors). All in all, the book is one which will probably stimulate many students and irritate some teachers.

R. F. RINEHART

Galois Lectures. By Jesse Douglas, Philip Franklin, Cassius Jackson Keyser, and Leopold Infeld. The Scripta Mathematica Library, Number 5. Scripta Mathematica, Yeshiva College, New York, 1941. 124 pages.

The lectures here collected were delivered at the Galois Institute of Mathematics at Long Island University in Brooklyn, New York. The first of them, "Survey of the Theory of Integration" by Jesse Douglas, is a clear, concise presentation of the fundamental definitions and essential properties of Riemann, Stieltjes, and Lebesgue integrals, with some remarks on the Denjoy integral. While avoiding excessive demands on the reader's mathematical background, the author deals clearly with many basic questions in a manner well adapted to the needs of a student of integration. The bibliography and specific references to sources for omitted details make the article a useful guide for a thorough study of the subject.

Philip Franklin's lecture is an exposition of the history and current status of the four-color problem. It contains a simplification of the problem to regular maps, a proof of the five-color theorem, a discussion of reducible configurations, and various specialized theorems on coloration. A few equivalent and inclusive formulations are presented. Franklin treats "perhaps the simplest unsolved problem of mathematics" in an interesting, lucid style, quite comprehensible to a non-specialist. He gives a bibliography which may tempt onward anyone who is lured by the fascinations of this mathematical will-o'-the-wisp.

The third lecture, by Cassius Jackson Keyser, is on "Charles Sanders Peirce as a Pioneer" and contains a very few samples of Peirce's vast and varied contributions to philosophic thought and especially to the development of logic. A brief biographical sketch is followed by sections telling of Peirce's pioneer work in connection with pragmatism, the theory of infinite classes, propositional functions, paradoxes, and the theory of relations. It is to be hoped that readers will be stimulated to consult the *Collected Papers* being published by the Harvard University Press.

The book closes with the text of a radio broadcast given under the auspices of the Galois Institute. It is in the form of a short dialogue between Leopold Infeld and a "clever pupil asking just the right questions" on the subject "The Fourth Dimension and Relativity." Remaining well within the college freshman level of difficulty, Infeld undertakes to cure the customary awe of the layman for those physicists and mathematicians who are supposed to have some intuitive knowledge of a four-dimensional universe.

S. S. CAIRNS

The Second Yearbook of Research and Statistical Methodology Books and Reviews. Edited by Oscar Krisen Buros. The Gryphon Press, Highland Park, New Jersey, 1941. 20+383 pages. $5.00.

In 1938, a set of excerpts from reviews of books on statistics and related fields appeared under the editorship of Professor Buros of Rutgers University

(see this MONTHLY, vol. 46, 1939, pp. 355–356). It consisted of 635 review excerpts from 131 journals. It was, on the whole, warmly received, and the editor has now published a greatly enlarged edition which is here under review. This edition, according to the preface, contains 1652 review excerpts from 283 journals. It is a large, excellently printed volume, very adequately indexed. Indeed, a list of the indices alone gives some idea of the thorough nature of the book; at the beginning, there is a list of the journals used, classified by fields, while at the end, there is a periodical directory and index, a publisher's directory and index, an index of titles, an index of names, and a classified index to books. The author states no less than eleven objectives of the book, or more properly, of the proposed series of similar volumes to appear every two years. These objectives may be summarized by stating that the primary idea is to elevate the standards of statistical literature by giving publicity to criticism written from a broad variety of viewpoints; a secondary aim is to elevate the standards of the criticism itself.

After this catalog of some of the more imposing features of the work under review, it will sound almost frivolous to say that, at least in the opinion of this reviewer, the book is really quite entertaining to read. This is undoubtedly a biased judgment, because several of this reviewer's own reviews are reprinted in juxtaposition to reviews on the same subject by better qualified reviewers, and the resulting comparisons were personally interesting and instructive. But the wide variety of opinions expressed on any one work, and the subtle light which they often throw upon their author's competence, would almost surely provide some entertainment even for a person not directly interested in statistics. Incidentally, the new edition covers an even greater variety of scientific fields than the older one did, which is really saying something. Perhaps this is in line with the definition of statistician which Dr. W. E. Deming likes to give in his lectures, at least by implication: a statistician is absolutely anyone who makes any measurements whatsoever.

The listings still do not include English reviews of foreign language books, an omission which was criticized adversely when it occurred in the earlier edition. In his preface, the editor expresses hope of remedying the situation in later editions, and also suggests two more major improvements toward which he is working. The first is to establish a noncritical abstracting service for the periodical literature on statistics, and the second is to publish original criticisms of articles and papers in the periodical literature. As far as mathematical statistics is concerned, the first of these services appears to be adequately supplied by *Mathematical Reviews*. As to the second part of the program, the plan sounds good at first hearing, but on second thought, the idea of supplementing each of the numerous "important" papers by several critical appraisals and (presumably, to be fair) a spirited rejoinder by the author, all sounds just a little appalling. Considering the amount of work required to establish and maintain *Mathematical Reviews*, one wonders if Professor Buros knows just what he is letting himself in for.

But to return to the present volume, we can say that the work as it stands

is both actually and potentially extremely valuable. Let us urge by all means that the series be continued. Every teacher of statistics courses will find the series indispensable, and it is now rather apparent that scientists engaged in a wide variety of activities will find these volumes useful.

J. H. CURTISS

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

A NIGHT WITH PROBABILITY

E. E. BLANCHE, Michigan State College

If you're looking for a program that will give pep to any club meeting, illustrate a number of mathematical principles, and, incidentally, show why gambling does not pay, then you might let probability do the job for you.

The arrangement of the program is rather simple; the equipment and supplies necessary are inexpensive; and the conduct of the various games at which members try their hand is so easy that even groups which have little mathematical background may understand the laws which "A Night With Probability" illustrates.

During the past year I have had occasion to direct this whirl with Lady Luck at our *Pi Mu Epsilon* chapter at the *University of Illinois*, and later at a number of university and church meetings. The entertainment has been received with such enthusiasm that it seems advisable to describe the entire procedure so that others may enjoy this educational and instructive evening of fun.

The entire program consisted of a number of simple games of chance, similar to those which may be found at county fairs, carnivals, amusement parks, and the "vice dens" of the big city. Before allowing the audience to act as the so-called "suckers" at the various gaming tables, I introduced the program with a twenty- to thirty-minute talk on various forms of amusement and games of chance which I have investigated during the past twelve years—investigation which was first undertaken as a reporter on the Passaic, N. J., *Daily News*, and later as a student and instructor in mathematics.

This sort of introduction acts as a stimulant, and puts the audience in the proper frame of mind. Some of the following games may be described and "debunked," or illustrated rather simply:

1. *The Numbers Racket.* This is very common in large cities and especially in the eastern states, notably New York and New Jersey. The player may select any number from 000 to 999, and place his bet of anything from one cent to one dollar with the local "bookie" (in general, a shady character). This is a daily

game. Based upon the winnings of the horses at some predetermined race track, a certain number is obtained, which is the winner that day. If one is unfortunate enough not to have placed a bet on that number (which is generally the case), he loses. On the other hand, if he has placed a bet on the winning number, then the "ring" conducting the "numbers" game will pay him 500 to 1 or 600 to 1, depending on the locality and the operating "ring." (The New York City area has recently cut its odds to 500 to 1.) If this is the case, then the player with a one-cent winner receives $5.00; but it is an unwritten law that the "bookie" will receive ten per cent of the winnings, so our player actually gets $4.50 for his penny. Actually, however, the odds against our friend winning are 999 to 1. If a wager of one cent were made on each number, totalling ten dollars, then the return would be only $4.50 on the single winning number. The gambling "ring" and the local "bookie" grow fat on the $5.50 which has been left in the till.

2. *Marked Cards*. These may sometimes be secured from local gamblers, local prosecutors' staffs after raids, or may be purchased from well known playing card manufacturers, some of whom do a thriving business in marked cards. (I shall be glad to furnish the names and addresses.) However, marked cards are expensive, the average cost being about five dollars a deck. The marking scheme, once learned, may be illustrated on a blackboard, and the cards may be distributed to the audience for examination.

3. *Carnival Wheel*. The ordinary wheel with twelve to twenty numbers on it sometimes has a brake similar to that on an automobile, and may be slowed down or stopped by the operator when he presses a foot pedal at some point quite distant from the wheel. The wheel may have a movable weight which may be adjusted by a touch of the operator to favor any section.

4. *Pitch-Till-You-Win*. This game allows you to toss hoops at prizes which are set up on cube or parallelopiped bases until a hoop covers the base. The bases of large prizes are such large cubes or parallelopipeds that it is nearly impossible to get a hoop completely over one on a toss from a distance of five to eight feet. Of course, the small prizes (ash trays, whistles, trinkets) have very tiny bases, which may be covered easily.

5. *The Fishing Game*. Here one scoops up a wooden fish from a circular trough of moving water. The moving fish are enticing and deceiving. The trough goes behind the curtain where the operator's confederate either removes or replaces winning fish at certain signals from the operator. Thus, while encouraging a prospect to try his luck, the operator can select a large prize winner; but when the prospect fishes, he always gets a small prize.

6. *Tossing Pennies* for one-inch squares which pay the tosser five or ten or even fifty cents for each penny lodged entirely within the square is a sure way to bankruptcy. Simple experimenting will show that several hundred pennies must be tossed before one is placed exactly inside a square.

7. *The Mouse Game* attracts a goodly crowd. A horizontal wheel is divided into sixty six-degree sections; at the circumference a small hole is drilled in each section; the sections are painted red, blue, green, yellow, orange, gold, and silver.

There are various numbers of sections of each color. The wheel is spun, and a mouse, covered by a small cup, is placed in the center. After a number of whirls, the cup is removed and the dizzy mouse, after regaining his equilibrium, scurries to one of the holes thinking he may find some food. The color of the section into which the mouse disappears determines the winner. Various odds are paid to the backers of the winning colors but these odds are tremendously in favor of the operator. The players are really paying for the privilege of watching the mouse run into a hole.

8. *Covering The Red Spot*, ten inches in diameter, with five small white circles is a task that requires minute exactness and much skill. The average person, without practice, has no idea of the single scheme necessary to blot out all the red color. This game may be mastered, however, by practice and development of skill.

9. The odds against the player in the *Rolling-Ball Game* are unimaginable. The player rolls six balls down an incline toward thirty-six holes, six each numbered one to six inclusive. The total score by the six balls which have fallen into various holes is the criterion by which winners are determined. However, only totals less than 13 or more than 29 will be winners. This presents a nice little problem of determining the probability of winning, namely, $2(924)/1{,}947{,}792 = .0008$ (eight times in ten thousand, approximately).

This type of introduction may be revised, or excluded, or supplemented by other games (I can supply hundreds more), according to the type of audience present or the amount of time allotted.

Following the introduction, a description of the various games to be played on "A Night with Probability" (which I have not yet described) may be given.

The following games have been employed with considerable success in illustrating probabilities and pointing out the truth of the statement "you can't win."

1. *Roulette*. A small or large roulette wheel is necessary for conducting this game, and may be purchased at toy shops, department stores, or borrowed from people who have them. The wheel consists of thirty-eight sections, numbered 0, 00, and 1 to 36. The rules of the game and the playing table may be painted with black ink on white sheets.

Bets of any size may be placed on any number 1 to 36. If the number turns up, the winner receives 35 to 1 or 36 to 1 for his wager. If his number does not turn up, he loses. In addition, the "House" or operator has the numbers 0 and 00, and when either appears, the "House" wins all wagers. Bets may also be placed on red or black, and even money is paid to winners; or sections of sets of numbers may be formed with suitable odds, as illustrated.

2. *Double-Your-Money-Quick*. This is a dice game, found any place that is frequented by dice players and gullible folk. Three dice are thrown by the operator of the game. A cardboard, with numbers 1, 2, 3, 4, 5, 6, is placed on a table. Bets may be placed on these numbers by the players. Now if one bets a unit on

No. 1, and the dice when rolled read 1-4-5, then the operator gives the player on 1 an additional unit. If there are players on 4 and 5, he gives each an additional unit. If there are players on 2, 3, and 6, he scoops up their bets. When all six numbers are covered by a unit each, and the operator tosses three different numbers on the three dice, he simply has to turn over the three units he wins on the losing numbers to the three people who backed the three winning numbers.

Now, when the dice are thrown 1-1-2, the operator gives two units to the player on 1, and one unit to the player on 2. If the board is filled with a single unit on each number, the operator will win a unit each from 3, 4, 5, and 6, and will then pay out two units to 1 and one unit to 2, leaving him a profit of one unit.

When the dice turn up 1-1-1, the operator pays the winner three additional units while he collects all other wagers. Under the same board conditions as above, he wins five units and pays out three.

Hence, whenever doubles or triples turn up on the dice, the operator takes a profit when the game is full. This game always wins for the operator when the players continue for some length of time.

3. *Match a Card.* The betting board has a set of cards exposed. The player simply picks out some card and places his bet upon it. Then, the operator selects a card from two decks shuffled together. If the number of the card is matched but not the suit, the bettor is paid 10 to 1. If both the number and suit are matched, the bettor receives 20 to 1. After the operator of the game has drawn eight or ten cards from the stack, the cards in the stack should be reshuffled so that the players do not have the advantage of having seen too many cards before placing their bets.

4. *Double or Nothing.* Here the persons taking part in the game do all the throwing of the dice while the one conducting the game simply pays the winners and collects from the losers. The idea behind the game is whether the tosser can throw doubles or triples with three dice. If the tosser does not throw doubles (two of a kind) or triples (three of a kind), he loses whatever he has bet. On the other hand, if he tosses doubles or triples, he wins from the operator as much as he has bet, *i.e.*, he doubles what he bet.

5. *Old-Fashioned Horse-Race.* This is a race between six horses, which may be made of wood by some member with a jig-saw. The horses are numbered and painted appropriate colors: (1) Spark Plug, (2) Barney Google, (3) Man O'War, (4) Whirlaway, (5) Flat Feet, (6) Zev. They are set at the starting-point of an oil cloth, ruled with ten spaces of five or six inches in length and two inches in width, in each of six lanes. Only six persons may play. The horses leave the starting-point and move one rectangle each time the number of the horse turns up in a toss of three dice by the operator. At the start of the race each player gives 5 units to the operator. If the first toss happens to be 1-2-2, then horse 1 moves up one rectangle, and horse 2 moves up two rectangles while the other horses remain at the post. Then the dice are tossed again, with various horses moving.

1. Roulette.

Odds 35 to 1		
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36		
1 2 3 4 5	Combination 5 to 1	6 7 8 9 10
11 12 13 14 15	16 17 18 19 20	21 22 23 24 25
Red (Even Money)	26 27 28 29 30	31 32 33 34 35
		Black (Even Money)

2. Dice.

1	2	3	4	5	6
Single No. Pays 1–1 Two of a Kind 2–1 Three of a Kind 3–1					

3. Match a Card.

Match No. or Card
Clubs A K Q J 10 9 8 7 6 5 4 3 2
Diamonds A K Q J 10 9 8 7 6 5 4 3 2
Hearts A K Q J 10 9 8 7 6 5 4 3 2
Spades A K Q J 10 9 8 7 6 5 4 3 2
Matched No. Pays 10–1 Matched Card Pays 20–1

4. Double or Nothing.

We Double Your Money When You Throw Doubles or Triples

5. Horse Race.

Starting Post

1	2	3	4	5	6
1	2	3	4	5	6

Finish Line

6. Twenty-One.

21 A Total of 19, 20, 21 Doubles Your Money. 2–1 For 21 With Two Cards Picture Cards—10 Points Aces Count 1 or 11 Others Face Value.

The procedure is repeated until one of the horses crosses the finish line. The player backing that horse is paid a win price of 12 units.

The winner is then out of the race, and the other five players continue to move their horses on each toss of the dice until another horse crosses the finish line. Second place wins eight units. The others continue until third place has been decided, paying 5 units. The other three horses are "also-rans," and their backers get nothing.

6. *Twenty-One*, a variation of the well known gambling game, is rather interesting. The conductor of the game is always the dealer of a deck of cards, but as a variation, the dealer does not take a hand. Each person who plays is dealt a single card, face down. Then, having looked at this card, the player may bet up to the limit set by the operator. Picture cards (Jacks, Queens, and Kings) count ten points each, while the Ace may be considered either one or eleven points. All numbered cards are counted their face value.

The player may draw as many cards as he wishes. If his total exceeds 21, he loses. If his total is 19, 20, or 21, he receives from the operator as much as he wagered. Moreover, if he makes 21 with only two cards, *i.e.*, an ace and a picture card or ten, the player is paid two units for every one wagered. Of course, another variation from the regular game is that each player will decide to draw cards until he has won with 19, 20, or 21, or has lost by exceeding 21.

This set of games may be supplemented by games of skill if it seems desirable. Some simple games of skill which are easily set up and which will not prove too expensive to the "House" are these:

1. *Ping Pong Bounce.* Six small glasses (fruit juice size) are placed in a triangle near a wall, with about an inch or two between each glass. A line is drawn about six feet in front of the triangle. Each contestant pays one unit for three ping pong balls, and tries to make them bounce into the glasses. Each time a ball bounces into a glass, the player gets five units. (Winners are rare.)

2. *Darts.* The player gets five darts for one unit of money, and wins a unit each time a dart lodges in the bull's eye.

3. *Milk-Bottle Drop.* The player buys three ping pong balls for a unit, and standing erect over a milk bottle, tries to drop the ping pong balls into the bottle from chin level. He is paid three units for each ball inside the bottle.

4. *Shuffleboard.* Five wooden wafers are given for one unit, and are shuffled down the board by the player. If four wafers lodge in the compartments at the end of the board, the player receives two units; if five wafers go into the compartments, the shuffler receives three units.

There are many other skill games which may be conducted, but the above have been tried with success.

Most of the equipment necessary for conducting the games may be purchased in five-and-dime and department stores. Wager boards similar to those illustrated may be made on large size cardboard with black or red marking crayon or poster inks. The horses may easily be cut, painted and numbered. The

dart game, roulette wheel, ping pong balls, shuffleboard and others may be borrowed from persons who have them, or purchased if there seems to be a likelihood of using them often.

The question of play-money is easily answered. Sheets of thin green and gold paper may be purchased at stationery stores (about ten cents a dozen sheets), and then cut to the appropriate currency size. The green paper may be considered one dollar bills while the gold paper may be considered ten dollar bills. There is no difficulty in recognizing the denomination. Or the numbers 1 and 10 may be written on the respective bills with marking crayon. (My wife and I made $2400 in half an hour from fifty cents worth of paper.)

Each person taking part in the evening's game should be given the equivalent of twenty dollars, one gold-back and ten green-backs. This small amount discourages wild splurging. The "House" should have in the neighborhood of several thousand dollars so that there will be no shortage of cash at any table during the evening.

After an hour or two of playing each person should return his money to the cashier. Prizes may be awarded to the persons returning the highest amount of money, *i.e.*, those who have been most successful at the games. In general, most people will lose their stake. Only a few will show any appreciable gain.

Besides providing plenty of laughs and fun for everyone taking part, "A Night With Probability" will teach those present why gambling does not pay.

Instead of the customary note printed by one of the New York daily newspapers at the bottom of each story on boxing, "Don't Bet on Fights," we should publicize the slogan, "Don't Bet on Anything."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 501. *Proposed by Daniel Arany, Budapest, Hungary.*

If A, B, C, I, J, X are six points on a conic, while L, M, N are points on the respective sides BC, CA, AB of the triangle ABC, and if further the three pencils $L(BXIJ)$, $M(CXIJ)$, $N(AXIJ)$ are projectively related, prove that the points L, M, N are collinear.

E 502. *Proposed by V. Thébault, San Sebastian, Spain.*

Find a number and its fourth power which together have nine digits, all different.

E 503. *Proposed by N. A. Court, University of Oklahoma.*

Through a point M lines are drawn meeting the pairs of opposite edges of a given tetrahedron in the pairs of points U, X; V, Y; W, Z. Prove that if M bisects each of the three segments UX, VY, WZ, it coincides with the centroid of the tetrahedron.

E 504. *Proposed by J. F. Kenney, University of Wisconsin at Milwaukee.*

If p and q are positive numbers with $p+q=1$, show that

$$\lim_{n\to\infty} \left(pe^{qt/\sqrt{npq}} + qe^{-pt/\sqrt{npq}}\right)^n = e^{t^2/2}.$$

E 505. *Proposed by H. T. R. Aude, Colgate University.*

How many different proper fractions when written in the ternary scale will be repeaters with not more than three digits in their repetends?

SOLUTIONS

E 463 [1941, 210]. *Proposed by N. A. Court, University of Oklahoma.*

Determine the locus of the trilinear pole of a given line with respect to the triangle along which a variable plane through the line cuts a given trihedral angle.

Solution by Howard Eves, Pittsburgh, Pa.

Let V be the vertex of the given trihedral angle, and ABC the section by a

plane passing through the given line p. Let P be the trilinear pole of p with respect to the triangle ABC, and let the plane VBC cut p in A'. Then the planes VAA', VAB, VAP, VAC form a harmonic set. Therefore P lies on a fixed plane through VA. Similarly, P lies on fixed planes through VB and VC. Hence the locus of P is a straight line through the vertex V.

Also solved by the proposer.

E 465 [1941, 210]. *Proposed by L. S. Johnston, University of Detroit.*

Without explicit use of the integral calculus, find the area enclosed by the curve $b^2y^2=(b+x)^2(a^2-x^2)$, where $b \geqq a > 0$.

Solution by Albert Furman, Marmion Military Academy, Aurora, Ill.

The equation written in the explicit form

$$y = \pm (1 + x/b)\sqrt{a^2 - x^2}$$

shows that the curve is a composite of the circle of radius a centered at the origin and the graph of the odd function $(x/b)\sqrt{a^2-x^2}$. On the right of the y-axis this last function adds to the area of the circle an amount equal to that subtracted from the circle on the left of this axis. Hence the required area is πa^2.

Also solved by E. F. Allen, Howard Eves, A. K. Waltz, and the proposer.

E 466 [1941, 266]. *Proposed by W. C. Rufus, Observatory of the University of Michigan.*

A is travelling in a restricted zone at two-thirds the speed limit, and B passes him, going twice as fast. Five minutes later a "speed cop," C, passes A and overtakes B. He spends two minutes giving out a ticket, then starts back at speed limit and meets A one mile back. Find at least one practicable solution. Discuss other possible solutions, limiting B's distance to five miles.

Solution by C. W. Trigg, Los Angeles City College.

If the speed limit is x m.p.h., it takes the "speed cop" t minutes travelling at y m.p.h. after he passes A to overtake B. If the distance from the point where C passes A to where C overtakes B be expressed in terms of B's travel and of C's travel, then

$$\frac{x}{18} + \frac{xt}{45} = \frac{yt}{60}.$$

Equating A's total travel plus one mile to B's total travel, we obtain

$$\frac{x}{18} + \frac{xt}{90} + \frac{x}{45} + \frac{2}{3} + 1 = \frac{x}{9} + \frac{xt}{45}.$$

Thus $t=3(50-x)/x$.

Speed limits are likely to be set only in multiples of 5, so we may tabulate the solutions as follows:

x (m.p.h.)	t (min.)	B's total travel	y (m.p.h.)
20	$4\frac{1}{2}$	$4\frac{2}{3}$	$41\frac{13}{17}$
25	3	$4\frac{1}{6}$	$61\frac{1}{3}$
30	2	$4\frac{2}{3}$	90

The most practicable solution would be the last of these, with A, B, and C travelling at 20, 40, and 90 m.p.h., respectively.

None of these solutions is essentially correct, for no provision has been made for the time and distance for deceleration after C overtakes B, nor for acceleration of C on the return journey.

Also solved by R. K. Allen, D. H. Browne, Howard Eves, Evelyn Hesseltine, E. P. Starke, and the proposer. Browne remarks that 90 m.p.h. is close to the record speed for a motorcycle; therefore he prefers the solution $x=20$, $y=41\frac{13}{17}$.

E 467 [1941, 266]. *Proposed by V. Thébault, San Sebastián, Spain.*

In a given triangle, show that the radical axes of the circumcircle with the respective circles whose diameters are the three medians, meet the corresponding sides in three collinear points.

Solution by Robin Robinson, Dartmouth College.

In the triangle ABC, let L, M, N be the midpoints of the sides, D, E, F the feet of the altitudes, and H the orthocenter. Since ADL is a right angle, the circle on AL as diameter passes through D. This circle, the circumcircle ABC, and the degenerate conic consisting of their radical axis AP and the line at infinity, all belong to the pencil of conics through A, P, and the circular points at infinity. By Desargues's involution theorem, the pairs of points in which the line BC cuts them belong to an involution. If AP meets BC at D', and A_∞ is the point at infinity on BC, these pairs of points are BC, LD, $A_\infty D'$. Thus

$$BCLA_\infty \barwedge CBDD';$$

and since BC, LA_∞ form a harmonic set, so do CB, DD'. Hence D', being the harmonic conjugate of D with respect to B and C, lies on the side EF of the complete quadrangle $AEFH$. Since the triangles ABC and DEF are in perspective from H, we deduce from Desargues's triangle theorem that D' and the analogous points on the other two sides are collinear.

Also solved by W. B. Clarke, H. A. DoBell, Howard Eves, P. D. Thomas, C. W. Trigg, and the proposer.

Editorial Note. The above proof shows that the line containing the three points is the trilinear polar of the orthocenter, namely, the *orthic axis* of the triangle ABC. Clearly, this is also the radical axis of the circumcircle and the nine-point circle. (For, B, C, E, F being concyclic, we have $D'B \cdot D'C = D'E \cdot D'F$.) Hence the orthic axis is perpendicular to the Euler line. In the special case of an equilateral triangle, it is the line at infinity (and the Euler line is indeterminate).

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS AND SOLUTION

4017. *Proposed by Norman Anning, University of Michigan.*

It is easy to show that $D(D^2+16)$ will annihilate $\cos^4 x+\sin^4 x$ and also $\cos^6 x+\sin^6 x$. Show in general that the same differential operator will annihilate similar expressions with exponents $4k$ and $4k+2$, where k is any positive integer.

4018. *Proposed by N. A. Court, University of Oklahoma.*

If four points taken in the four faces of a tetrahedron are collinear, their trilinear polars for the respective faces cannot be (*a*) coplanar, or (*b*) hyperbolic.

4019. *Proposed by Robin Robinson, Dartmouth College.*

Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

4020. *Proposed by V. Thébault, Le Mans, France.*

With three consecutive odd digits form a number of five digits whose square has ten distinct digits.

SOLUTIONS

3957 [1940, 245]. *Proposed by Otto Dunkel, Washington University.*

Given a triangle ABC with angles $A<B<C$, show that there are precisely one, two, three straight line segments which bisect both its perimeter and area according as

$$1-\frac{\sin A}{\sin B}\gtreqless 2\tan^2(A/2)\tan^2(B/2),$$

where we may replace B by C. If $B=C$, there are one, two, three such segments, according as $A\gtreqless A_0$, where $\sin(A_0/2)=\sqrt{2}-1$.

Solution by E. P. Starke, Rutgers University.

Let P be any one of the three angles, p the opposite side, and q, r the adjacent sides. Let a segment cut off a length θq on side q and $\theta' r$ on side r measured from P, with θ, $\theta'<1$. If the segment bisects the area $\frac{1}{2}qr\sin P$, we must have $\theta\theta'=\frac{1}{2}$.

If the perimeter is also bisected, we have $\theta q+\theta' r=\frac{1}{2}(p+q+r)$ or $2\theta q+r/\theta=p+q+r$. Putting

$$f(\theta) = 2\theta^2 q + r - \theta(p + q + r),$$

we require $f(\theta)=0$, $\frac{1}{2}<\theta<1$. Now $f(\frac{1}{2})=\frac{1}{2}(r-p)$ and $f(1)=q-p$. If $f'(\theta_0)=0$, then $\theta_0=(p+q+r)/4q$ and $f(\theta_0)=\{8qr-(p+q+r)^2\}/8q$. Thus if p is taken as b, $f(\frac{1}{2})$ and $f(1)$ are of opposite signs and there is one segment meeting sides a and c. If p is c, then $f(\frac{1}{2}), f(1), f(\theta_0)$ are all negative,* and there is no segment meeting a and b. Corresponding to $p=a$, $f(\frac{1}{2})$ and $f(1)$ are both positive and there are no segments, one segment, or two segments meeting b and c, according as $f(\theta_0)\gtreqless 0$, *i.e.*, according as $(a+b+c)^2\gtreqless 8bc$†. This can be transformed into the proposed condition by the following steps:

$$\begin{aligned}
(a + b + c)^2(-b - a) &\gtreqless 8bc(-b - a) = -8bc(a + b + c) + 8bc^2,\\
(a + b + c)^2(b - a) &\gtreqless 2b(a + b + c)^2 - 8bc(a + b + c) + 8bc^2\\
&= 2b(a + b + c - 2c)^2,\\
\frac{b - a}{b} &\gtreqless \frac{2(a + b - c)^2}{(a + b + c)^2}\\
&= 2\left\{\frac{(a - b + c)(a + b - c)}{(a + b + c)(b + c - a)}\right\}\left\{\frac{(b + c - a)(b - c + a)}{(a + b + c)(a - b + c)}\right\},\\
\frac{\sin B - \sin A}{\sin B} &\gtreqless 2\tan^2(A/2)\tan^2(B/2).
\end{aligned}$$

If however, $B=C$, so that $\sin B=\cos(A/2)$, then $(a+b+c)^2\gtreqless 8bc$ gives $(a+2b)^2\gtreqless 8b^2$ or $(\sin A+2\sin B)^2\gtreqless 8\sin^2 B$ or

$$\sin A + 2\cos(A/2) \gtreqless 2\sqrt{2}\cos(A/2).$$

When $\sin A$ is replaced by $2\cdot\sin(A/2)\cos(A/2)$, this becomes

$$\sin(A/2) \gtreqless \sqrt{2} - 1 = \sin(A_0/2).$$

whence $A\gtreqless A_0$.

Editorial Note. The last part, for which $B=C$, requires an examination of the values of θ excluded in the first part. The case $A=B=C$ is easily disposed of, since then each of the three angles is greater than A_0 and there are precisely three segments.

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* $p>q$ implies $(p+q+r)^2>(2q+r)^2=(2q-r)^2+8qr\geqq 8qr$.

† Since this relation involves b and c symmetrically, these letters may be interchanged in what follows.

A theorem useful for the construction of the required segments may be stated as follows. If a transversal of triangle PQR satisfies two of the following requirements, it satisfies the third; namely, it passes through the incenter I, it bisects the perimeter, it bisects the area. If we consider only the requirement that the transversal bisects the perimeter $2s$, then this may be stated in the following manner. On PR lay off the segment PS_q in the direction PR and with the length s; similarly, lay off on PQ the segment PS_r. Then PI passes through N, the midpoint of S_rS_q, and the perpendicular to PN at its midpoint V bisects PS_q, PS_r in V_q, V_r. Now take Q' on the straight line of PR, and R' on PQ, so that $PR'=Q'S_q$ or what is the same, $PR'+PQ'=s$. The two ranges of points R', Q' are projective, and hence $R'Q'$ is tangent to a conic which is tangent to the line at infinity and also to PQ, PR at S_r, S_q. The conic is therefore a parabola tangent to V_rV_q at V the vertex, and with the axis along VN. The perpendicular to PR at V_q cuts VN in F, the focus. There is no tangent through I if I lies outside PV, and just one if it is at V. If I lies within PV, the circle on IF as diameter cuts V_rV_q in two points each of which joined to I by a straight line gives a tangent to the parabola. This construction gives the desired segments $R'Q'$ of the problem, if there are any such for the two considered sides PQ, PR.

We may also consider the single condition that $R'Q'$ bisects the area S of PQR. Then $PQ'\cdot PR'=qr/2=k^2$; the points R', Q' describe projective ranges on PQ, PR, and the line $R'Q'$ is tangent to a conic. Since the conic is tangent to PQ, PR at infinity, it must be a hyperbola with its transverse axis along PI. On PR lay off the segment PV_q' in the direction PR and with the length k, and determine V_r' on PQ similarly. Then the hyperbola is tangent to $V_r'V_q'$ at its midpoint V', and the focus F on the extension of PV' is easily constructed. A construction for the tangents through I for this branch, if there are any such, may be easily obtained. There are other geometric results relating to I and the two conics.

The desired conditions of the problem may be found by considering the rotation of a transversal $R'Q'$ through I with R' on PQ and limited to that segment, while Q' is on and limited to PR. When Q' is at R, let the end R' be at R_i; when R' is at Q, let Q' be at Q_i. Let the transversal through I perpendicular to PI cut PQ, PR in I_r, I_q; then the order of points on PR is P, Q_i, I_q, R, and similarly, for PQ the order is P, R_i, I_r, Q. Let the parallel to PQ through I_q cut R_iR, $R'Q'$, QQ_i in R_i', R'', Q_i'; then R_i' is inside PQR and Q_i' is outside. Denoting the area of a figure by its vertices, we find that $PR'Q'=PR_iR-R''R_i'RQ'$ when Q' is on segment RI_q; and if Q' is on I_qQ_i, then $PR'Q'=PI_rI_q+R''Q'I_q$. Hence, as Q' moves from R to I_q, the area $PR'Q'$ steadily decreases from PR_iR to PI_rI_q, and then steadily increases from PI_rI_q to PQQ_i, as Q' moves from I_q to Q_i. The ratio $PR_iR/S\gtreqless 1/2$, according as $q\gtreqless p$, since RR_i is an internal bisector of angle R. We now consider only the cases where Q' is a desired point within the segment RQ_i; that is, we consider non-vertex positions for the sides of lengths q, r. We have at once three results:

$$(1)\qquad (q, r):\quad \begin{array}{l}\text{case 1, } q \leqq p \text{ and } r \leqq p, \ (0);\\ \text{case 2, } q \leqq p \text{ and } r > p, \ (1);\\ \text{case 3, } q > p \text{ and } r \leqq p, \ (1);\end{array}$$

where the number in the last parentheses denotes the number of non-vertex positions. There remains the case 4, where $q>p$ and $r>p$,and we must now consider the area of PI_rI_q, or of its complement I_qI_rQR, and we select the latter. We have

$$(2)\qquad \begin{aligned}\frac{I_qI_rQR}{S} &= \frac{RR_iQ}{S}+\frac{I_qR_i'R}{S} = \frac{p}{p+q}+\frac{q}{p+q}\left(\frac{I_qR}{q}\right)^2\\ &= \frac{p}{p+q}+\frac{q}{p+q}\tan^2\frac{Q}{2}\tan^2\frac{P}{2},\end{aligned}$$

where we have used the relations

$$(3)\qquad q=\rho\frac{\cos\dfrac{Q}{2}}{\sin\dfrac{P}{2}\sin\dfrac{R}{2}}, \qquad I_qR=\frac{\sin\dfrac{Q}{2}}{\sin\dfrac{R}{2}}\left(\frac{\rho}{\cos\dfrac{P}{2}}\right), \qquad \rho=\text{inradius}.$$

Hence for case 4 there are 0, 1, 2 non-vertex positions according as the right side of (2) is less than, equal to, or greater than 1/2, or what is the same

$$(4)\qquad 1-\frac{\sin P}{\sin Q}\gtreqless 2\tan^2\frac{P}{2}\tan^2\frac{Q}{2}.$$

This leads easily to the results of the problem when no two angles of the triangle are equal. For the case $Q=R$, the consideration of case 4 is much simpler. Let PI cut QR in M; then

$$(5)\qquad \frac{PI_rI_q}{S}=\left(\frac{PI}{PM}\right)^2=\left(\frac{1}{1+\dfrac{\rho}{PI}}\right)^2=\left(\frac{1}{1+\sin\dfrac{P}{2}}\right)^2,$$

and the ratio on the left is greater than, equal to, or less than 1/2 according as $\sin(P/2)\gtreqless\sqrt{2}-1=\sin(A_0/2)$. Considering the three pairs of sides of ABC, we obtain the results given in the problem.

We easily find directly from the figure that $PI_rI_q/S=rq/s^2$, and it will be readily found that this leads in case 4 to the inequalities used by Starke. It will be observed that this ratio may be also written $2(PV_q')^2/(PS_q)^2$, thus connecting it with the two conics in the first part of this note.

A solution by Kwan Chao-chih, Yenching University, was received after the preparation of the above for printing. In this solution the general triangle

$A_1A_2A_3$ was considered first with a required segment cutting the two sides through A_i in the points P_i and Q_i, $A_iP_i=x_i$, $A_iQ_i=y_i$. Then we must have $x_i+y_i=s$ and $2x_iy_i=a_ja_k$. Conditions are then easily set down for two, one, no distinct segments for this pair of sides. Applying this first to the case where no two sides are equal, and then to the case of two equal sides, results were obtained as in the problem. As in Starke's solution, the treatment was mainly an algebraic study of the corresponding quadratic expression.

3958 [1940, 322]. *Proposed by Michael Goldberg, Washington, D. C.*

What is the probability P_n of making a sequence of at least r throws of a number in n throws of a die when the probability of throwing that number in a single trial is p?

Solution by C. Eisenhart, University of Wisconsin.

The probability P_n is given by the coefficient of y^n in the power series expansion of

$$p^r y^r \cdot \frac{(1-py)}{(1-y)^2+(1-p)p^r y^{r+1}(1-y)}. \tag{1}$$

The contrary probability $1-P_n$ is the coefficient of y^n in the expansion of

$$\frac{1-p^r y^r}{1-y+(1-p)p^r y^{r+1}}. \tag{2}$$

This is Proposition XLVI in the 3rd edition (1878) of William Allen Whitworth's *Choice and Chance*—it is Proposition LIII of the 5th edition which is available in reprint form from G. E. Stechert & Co., New York City—expressed in the notation of the present problem. Whitworth's proof is quite direct, and, being readily available, will not be reproduced here.

Since we desire P_n, we must subtract (2) from the function $(1-y)^{-1}$ in the expansion of which the coefficient of y^n is always unity. This gives P_n as the coefficient of y^n in the expansion of

$$\frac{1}{1-y}-\frac{1-p^r y^r}{1-y+(1-p)p^r y^{r+1}}, \tag{3}$$

which, being expressed with a common denominator, gives (1).

That P_n calculated as specified from (1) or (3) satisfies the "boundary conditions" of the problem can be seen as follows:

(*a*) When $r=0$, (3) reduces to $(1-y)^{-1}$, whence $P_n=1$ for all n as it should.

(*b*) When $r=1$, (3) becomes

$$(1-y)^{-1}-(1-[1-p]y)^{-1}=\sum_{n=0}^{\infty}(1-[1-p]^n)y^n,$$

giving $P_n=1-(1-p)^n$ as it should.

(*c*) When $r=n$, (1) may be written

$$p^n y^n\left(1+\sum_{t=1}^{\infty} a_t y^t\right), \tag{4}$$

since the fractional part of (1) has the value 1 when $y=0$. From this it is clear that $P_n=p^n$ in this case.

(*d*) When $r>n$, $P_n=0$ as is clear from (4) with n replaced by $r>n$.

For other values of r it is not so easy to obtain an expression for P_n valid for all n, but P_n can be obtained from (1), or arrangements of it, by various artifices and perseverance with division or differentiation. For example, it is not difficult to see that $P_n=p^{n-1}(2-p)$ for $r=n-1$.

Later addition. This problem was originally proposed and solved by De Moivre. A complete solution with discussion of approximations for large n is given on pp. 77–84 of Uspensky's *Introduction to Mathematical Probability.*

Solved also by R. E. Moritz.

Editorial Note. R. E. Moritz gave a development of a generating function for $Q_n=1-P_n$ in the form

$$\frac{p^{n+1}(1-x^r)}{p-x+(1-p)x^{r+1}},$$

and then, by expansion of this function in a power series in x, found Q_n as the coefficient of x^n. This derivation is long and complicated.

It is possible to obtain the desired probability without the use of a generating function, and to simplify the work we define as follows a related probability $p_2(n)$ upon which the desired one depends:

In a sequence of n trials of the same experiment, the success S of each has the same probability μ. The probability of beginning the sequence with at least r successes with no subsequent failure F followed by as many as r successes is denoted by $p_2(n)$.

We derive first the difference equation for $p_2(n)$. If for each favorable case for $p_2(n)$ we follow it by an F, we get all the favorable cases for $p_2(n+1)$ ending in an F. If for each favorable case in $p_2(n-j)$, $1\leqq j\leqq r-1$, we follow it by an F and j of the S's, we get all the favorable cases in $p_2(n+1)$ ending in j of the S's. The only favorable case in $p_2(n+1)$ ending in r of the S's is the one for which each of the $n+1$ results is an S. Hence we have

$$\begin{aligned} p_2(n+1) = (1-\mu)[p_2(n) + \mu p_2(n-1) + \mu^2 p_2(n-2) + \cdots \\ + \mu^{r-1} p_2(n-r+1)] + \mu^{n+1}. \end{aligned} \tag{1}$$

We have obviously $p_2(n)=0$, $1\leqq n\leqq r-1$; $p_2(r)=\mu^r$; and we then find easily from (1) that $p_2(0)=0$. Hence, we may take as initial conditions for (1) that $p_2(n)=0$, $0\leqq n\leqq r-1$. If we write the equation (1) replacing n by $n-1$, multiply each term of it by μ, and subtract from (1), we get a simpler equation of order one unit

higher. We write this equation in the form

$$(2) \qquad p_2(n+1) - p_2(n) = \Delta p_2(n) = -\lambda p_2(n-r), \qquad \lambda = (1-\mu)\mu^r,$$

with the initial conditions $p_2(n)=0$, $0 \leqq n \leqq r-1$; $p_2(r)=\mu^r$.

Equation (2) may be derived directly. If we follow each favorable case in $p_2(n)$ by either an F or an S, we obtain all favorable cases in $p_2(n+1)$ and some unfavorable cases. The latter are those of $p_2(n)$ which have at the end an F followed by precisely $r-1$ of the S's, so that the additional S makes it unfavorable. The probability of this latter is $(1-\mu)\mu^r\, p_2(n-r)$, and then (2) follows immediately.

The formula for $p_2(n)$ will be found by successive steps and in each step we use the known formula $\Delta x^{(k)} = kx^{(k-1)}$, where x is any number and $x^{(k)} = x(x-1)\cdots(x-k+1)$, $x^{(0)}=1$. First we have $\Delta p_2(n)=0$, $r \leqq n \leqq 2r-1$; hence $p_2(n)=\mu^r$ for $r \leqq n \leqq 2r$. Next we have $\Delta p_2(n) = -\lambda\mu^r$, $2r \leqq n \leqq 3r$; and hence $p_2(n) = \mu^r[1-\lambda(n-2r)]$ for $2r \leqq n \leqq 3r+1$, where the constant of summation has been determined. For the third step we have $\Delta p_2(n) = \mu^r[-\lambda+\lambda^2(n-3r)]$, $3r \leqq n \leqq 4r+1$. Then

$$p_2(n) = \mu^r[1 - \lambda(n-2r) + \lambda^2(n-3r)^{(2)}/2], \qquad 3r \leqq n \leqq 4r+2,$$

where we have naturally preserved the terms already found, and the constant of summation is properly determined as we now show. If we set in this last expression $n=3r$, $3r+1$, we get results already determined in the second step, and thus no other constant is to be added. This suffices to indicate that

$$(3) \qquad p_2(n) = \mu^r \sum_{j=1}^{i} (-1)^{j-1} \frac{(n-jr)^{(j-1)}}{(j-1)!} \lambda^{j-1}, \qquad ir \leqq n \leqq (i+1)r+i-1, \quad (i = 1, 2, \cdots).$$

We now prove that this is correct by assuming that (3) has been proved for a given $i \geqq 1$ and then showing that it is also true for $i+1$. By hypothesis,

$$\Delta p_2(n) = \mu^r \sum_{j=1}^{i} (-1)^j \frac{[n-(j+1)r]^{(j-1)}}{(j-1)!} \lambda^j, \quad (i+1)r \leqq n \leqq (i+2)r+i-1.$$

Then by the method of the above steps we have, after slight changes in the limits of the summation,

$$p_2(n) = \mu^r \sum_{j=1}^{i+1} (-1)^{j-1} \frac{(n-jr)^{(j-1)}}{(j-1)!} \lambda^{j-1}, \qquad (i+1)r \leqq n \leqq (i+2)r+i.$$

No further constant is to be added, since for $(i+1)r \leqq n \leqq (i+1)r+i-1$, the term for $j=i+1$ in this last expression vanishes and the expression is then the one of the hypothesis. Also, this last interval for n is common to the hypothesis interval and the derived interval. The formula (3) is now completely proved. We may avoid the overlapping of consecutive intervals by writing (3) in a form more convenient for computation, namely,

$$(4) \qquad p_2(n) = \mu^r \sum_{j=1}^{i} (-1)^{j-1} {}_{n-jr}C_{j-1} \lambda^{j-1}, \qquad i = \left[\frac{n+1}{r+1}\right],$$

where ${}_mC_t$ denotes a binomial coefficient.

We now define $p_1(n)$ as the probability that in n trials there is at least one F followed by at least r of the S's. Then $p_3(n) = p_1(n) + p_2(n)$ is the probability desired in the present problem. Also, $q_1(n) = 1 - p_1(n)$ is the probability that in n trials no F is followed by as many as r of the S's. If we consider each favorable case for $q_1(n)$ following r of the S's, we get the favorable cases for $p_2(n+r)$. Hence $p_2(n+r) = \mu^r q_1(n)$, and then we have

$$(5) \qquad p_3(n) = 1 + p_2(n) - \mu^{-r} p_2(n+r);$$

this completes the solution of the present problem, where $\mu = p$ and $p_3(n) = P_n$.

We now give an evaluation of $p_2(n)$ which requires neither the generating function nor the difference equation. Consider a straight line divided into n spaces for receiving a letter in each space. In each of the first r spaces we place an S, and then j blocks each consisting of an F followed by r of the letters S, the blocks occupying $j(r+1)$ spaces. The number of ways the j blocks may be placed on the $n-r$ spaces is ${}_{n-(j+1)r}C_j$, where $j \leqq i-1$, $i = [(n+1)/(r+1)]$. The remaining $n-r-j(r+1)$ free spaces are then filled in any manner by the letters S, F. This gives a representation of n experiments with at least r of the S's at the start and with at least j of the F's each followed by at least r of the S's. The probability of n experiments resulting this way is

$$(6) \qquad \nu_j = \mu^r \, {}_{n-(j+1)r}C_j \lambda^j.$$

We shall prove that

$$(7) \qquad \sum_{j=0}^{i-1} (-1)^j \nu_j = p_2(n) = \mu^r \sum_{j=1}^{i} (-1)^{j-1} \, {}_{n-jr}C_{j-1} \lambda^{j-1}.$$

Consider the probability of a definite arrangement of this kind, where there are precisely u blocks in given positions. Then this probability is counted ${}_uC_j$ times in ν_j, $j \leqq u$; and the number of times it is counted in the sum (7) is

$$\sum_{j=0}^{u} (-1)^j \, {}_uC_j = 0.$$

Hence in (7) the probability of an F followed by at least r of the S's is eliminated, and the proof is complete.

The generating function is desirable in the computation of an approximating expression for $p_3(n)$ when n is large. In this case it is more convenient to obtain such a function for $p_2(n)$. The auxiliary algebraic equation for (2) is $x^{r+1} - x^r + \lambda = 0$, and we set

$$\frac{\phi(y)}{1-y+\lambda y^{r+1}} = \sum_{n=0}^{\infty} p_2(n)\, y^n,$$

and then find easily that $\phi(y) = \mu^r y^r$.

The special case of $\mu = 1/2$ is considered in the solution of 3046 [1924, 403], where the difference equation for $p_3(n)$ is solved in a manner similar to the above. For $n = 50$, $r = 5$, it is there stated that $p_3(50) = .55188$. In an article entitled *Two types of probabilities and their difference equations*, by Otto Dunkel (Washington University Studies, Scientific Series, No. 2, 1925, pp. 119–136), there is a small table $\mu = 1/2$, $r = 5$, for $p_1(n)$, $p_2(n)$, $p_3(n)$ with $n = 5, 10, 15, \cdots, 50$. In this MONTHLY in Dunkel's article on *Solutions of a probability difference equation* (vol. 32, 1925, pp. 354–370), there is given the development of the generating function for $P_2(n) = 2^n p_2(n)$, the computation of approximate formulas for large values of n, and a method for computing the root of the associated algebraic equation which is used in the approximate formula. In the solution of 3363 [1930, 319] the case of μ equal to the reciprocal of an integer greater than unity is considered. A discussion of the absolute values of the roots of the auxiliary algebraic equation is given in the solution of 3162 [1929, 105].

3965 [1940, 491]. *Proposed by H. S. Wall, Northwestern University.*

Show that for a real or complex x, $|x| \leqq 1$,

$$\frac{|x|}{1+|x|} \leqq |\log(1+x)| \leqq \frac{|x|\,(1+|x|)}{|1+x|}.$$

I. *Solution by S. E. Warschawski, Washington University.*

To prove the right-hand inequality we make use of the Taylor series for $\log(1+z)$ which represents the principal value of the logarithm for $|z| < 1$. We write

$$\begin{aligned}(1+z)\log(1+z) &= (1+z)\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right)\\ &= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\\ &\quad + z^2 - \frac{z^3}{2} + \frac{z^4}{3} - \cdots\\ &= z + \frac{z^2}{2} - \frac{z^3}{2\cdot 3} + \frac{z^4}{3\cdot 4} - \cdots\\ &= z\left\{1 + z\left[\frac{1}{2} - \frac{z}{2\cdot 3} + \frac{z^2}{3\cdot 4} - \cdots\right]\right\}.\end{aligned}$$

Hence, since $|z|<1$, we have

$$|(1+z)\log(1+z)| \leqq |z|\left\{1+|z|\left[\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\right]\right\}$$
$$= |z|\left\{1+|z|\left[\tfrac{1}{2}+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\cdots\right]\right\}$$
$$= |z|\left\{1+|z|\right\},$$

or

$$|\log(1+z)| \leqq \frac{|z|(1+|z|)}{|1+z|}.$$

We now consider the left-hand inequality. The function

$$w = \log(1+z) = \int_0^z \frac{dt}{1+t}$$

(which is the principal value of the logarithm) is regular in the region D obtained from the z-plane by cutting it along the negative real axis from -1 to $-\infty$. The map of D in the w-plane is the strip $-\pi<I(w)<\pi$. Take a fixed point z_0 in D; let its map in the w-plane be $w_0=\log(1+z_0)$. Connect w_0 with the origin in the w-plane by the straight line segment $\overline{Ow_0}$. The map of this line in the z-plane by means of the inverse function $z=e^w-1$ is a certain curve C which connects $z=0$ with $z=z_0$. The length of the line $\overline{Ow_0}$ may now be expressed as

$$|w_0| = \int_C \frac{|dt|}{|1+t|}.$$

Let C' be the arc obtained when C is traversed from $z=0$ to its first point of intersection with the circle $|z|=|z_0|$. Then certainly

$$|w_0| \geqq \int_{C'} \frac{|dt|}{|1+t|}.$$

Since $|1+t|\leqq 1+|t|\leqq 1+|z_0|$ on C', we have

$$|w_0| \geqq \int_{C'} \frac{|dt|}{1+|z_0|} = \frac{1}{1+|z_0|}\int_{C'} |dt| \geqq \frac{|z_0|}{1+|z_0|}.$$

The left-hand inequality has now been proved for the whole z-plane.

II. *Solution by A. M. Gleason, New Haven, Conn.*

(1) The upper inequality. Since

$$\frac{\log(1+x)}{x} = \int_0^1 \frac{dt}{1+tx},$$

$$\left|\frac{\log(1+x)}{x}\right| \leqq \int_0^1 \frac{dt}{|1+tx|} \leqq \max_{0\leqq t\leqq 1} \frac{1}{|1+tx|} \equiv M.$$

(*i*) If x is in the right half plane, then $M=1\leqq(1+|x|)/|1+x|$.

(*ii*) If $|x+\frac{1}{2}|\leqq\frac{1}{2}$, then $M=1/|1+x|\leqq(1+|x|)/|1+x|$.

(*iii*) If x falls within neither of these regions, then if $R(x)\geqq -1$, the triangle OAP, $\{O\ (0,0), A\ (-1,0), P(x)\}$ has all acute angles, and hence the sum of the sines of the angles is greater than 2. Using the law of sines, we have

$$M = \frac{1}{\sin O} = \frac{OA+OP}{AP}\,\frac{1}{\sin P+\sin A} \leqq \frac{OA+OP}{AP} = \frac{1+|x|}{|1+x|}.$$

This establishes the upper inequality throughout the half-plane $R(x)\geqq -1$.

(2) The lower inequality. This is apparent graphically along the real axis from -1 to $+\infty$. Along the unit circle, we have

$$|\log(1+e^{i\theta})| = |\log 2\cos\tfrac{1}{2}\theta + \tfrac{1}{2}\theta i| \geqq \tfrac{1}{2},$$

since if $\theta\geqq 1$ and if $\theta<1<60°$, then $\log 2\cos\frac{1}{2}\theta > \log\sqrt{3} > \frac{1}{2}$.

Consider a path along which $|1+x|$ is constant, extending from the real axis to the unit circle. The functions $|\log(1+x)|$ and $|x|/(1+|x|)$ are both continuous along this path, and differentiable within the interior. Using $\phi=\arg(1+x)$ as independent variable, we have

$$|x|^2 = |1+x|^2 + 1 - 2|1+x|\cos\phi, \qquad 2|x|\frac{d|x|}{d\phi} = 2|1+x|\sin\phi.$$

Then

$$\begin{aligned}\frac{d}{d\phi}\left\{|\log(1+x)|^2 - \left(\frac{|x|}{1+|x|}\right)^2\right\} &= \frac{d}{d\phi}\left\{\log^2|1+x| + \phi^2 - \frac{|x|^2}{(1+|x|)^2}\right\}\\ &= 2\phi - 2\frac{|x|}{1+|x|}\,\frac{1}{1+|x|)^2}\,\frac{d|x|}{d\phi}\\ &= 2\phi - 2\frac{|1+x|}{1+|x|}\,\frac{1}{(1+|x|)^2}\sin\phi,\end{aligned}$$

which cannot vanish for $\phi\neq 0$. Applying Rolle's theorem, and remembering that the inequality has been shown at either end-point, we know that if it failed anywhere in the middle the derivative must vanish somewhere in the interior. This establishes the lower inequality.

Solved also by E. S. Pondiczery.

Editorial Note. Pondiczery's solution considered $|x|<1$, and then extended the results to $|x|=1$, excluding $x=-1$. His proof of the right-hand side of the

inequality is similar to the one in solution I. For the left-hand side, after setting $x=|x|e^{i\theta}$, we have

$$\log(1+x)=\int_0^{|x|}\frac{e^{i\theta}dr}{1+re^{i\theta}},$$

$$|\log(1+x)|=\left|\int_0^{|x|}\frac{dr}{1+re^{i\theta}}\right|\geqq\left|\Re\left\{\int_0^{|x|}\frac{dr}{1+re^{i\theta}}\right\}\right|$$

$$\geqq\int_0^{|x|}\frac{dr}{1+|x|}=\frac{|x|}{1+|x|}.$$

The last parts of the reduction follow from the inequalities

$$\Re\left\{\frac{1}{1+re^{i\theta}}\right\}=\frac{1+r\cos\theta}{1+r^2+2r\cos\theta}\geqq\frac{1}{1+r}\geqq\frac{1}{1+|x|}.$$

The trigonometric theorem used in solution II may be obtained from the identity $\sin^2A+\sin^2B+\sin^2C=2(1+\cos A\ \cos B\ \cos C)$, where $A+B+C=\pi$. For an actual acute angled triangle the sum of squares of the sines is greater than 2, and this must also be true for the sum of the sines.

In the last part of this solution it follows from the final steps that $dy/d\phi$, where y is the expression in brackets, is zero if $\phi=0$ and positive if $\phi>0$ when $|1+x|$ is constant. When $\phi=0$, $y\geqq 0$, and hence for $\phi>0$ we must have $y>0$. This proves the left-hand side for the whole upper half-plane without use of the unit circle. Then for the lower half-plane it must also be true by considerations of symmetry with respect to the axis of reals.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Crawfordsville, May 1–2, 1942
IOWA, Mt. Pleasant, April 17–18, 1942
KANSAS, Hays, March 27–28, 1942
KENTUCKY, Lexington, April 11, 1942
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 6–7, 1942
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 1942
METROPOLITAN NEW YORK, New York, April 18, 1942
MICHIGAN
MINNESOTA
MISSOURI, Kansas City, April 17, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, Berkeley, Jan. 31, 1942
OHIO, Columbus, April 2, 1942
OKLAHOMA, Oklahoma City, Feb. 13, 1942
PHILADELPHIA, Philadelphia, Nov., 28, 1942
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SOUTHWESTERN, State College, N. M., April 27–28, 1942
TEXAS, Lubbock, April 3–4, 1942
UPPER NEW YORK STATE, Rochester, May 2, 1942
WISCONSIN, Oshkosh, May 2, 1942

Twenty-fifth Summer Meeting, Ithaca, New York, September 7–9, 1942.

SELECTIVE SERVICE FOR MATHEMATICIANS

The object of this open letter is to give to interested mathematicians a report concerning developments in the problem of the use of mathematicians of the draft age.

At a July meeting of the Roster of Scientific and Specialized Personnel in Washington these matters were discussed at length and a number of those present, including the writer, urged that the Roster is the body best equipped to take up the problem of the proper use of scientists. It should be understood at the beginning of this letter that the Roster has no power to defer scientists, but is concerned only with making recommendations as to whether or not a man be regarded as "necessary" in the sense of the law.

In order to make proper recommendations it was necessary that the Roster obtain considerable additional information concerning men of draft age on its lists. The matter of procedure has now been systematized. Any man who is on the Roster and who finds himself likely to be called may write the Roster for a standard questionnaire. Among other things this questionnaire seeks to find out the man's present occupation and scientific status and the names of references. These references and the man's employer are then sent questionnaires calculated to assist in determining whether the man may be regarded as "necessary."

The Roster then calls in men in the various scientific fields to review the assembled data and advise the Roster on each individual case. The mathematicians were the first so treated, and Murnaghan, Hotelling, and Morse recently served as consultants for the Roster. The Roster had assembled excellent data bearing on 300 cases, all of which was carefully reviewed by each of the three mathematicians. In general we tried to follow the resolutions which were voted at the recent Chicago meeting of the mathematicians and are to be published in the *Bulletin of the American Mathematical Society*.

The Roster makes its recommendations to the Selective Service Headquarters, which in turn *may* pass on the recommendations to the local boards. The local boards are free to follow the recommendations or not.

In case a man is inducted, the Roster acts in an advisory capacity to the personnel department of the Adjutant General's office, offering information which will aid the personnel officers in properly using the scientists on the Roster.

In addition to making recommendations as to individual men, the Roster also seeks information concerning shortages in various scientific fields and passes on this information to the Selective Service Headquarters and the Labor Office. Our Committee on the Supply and Demand for Mathematicians will materially aid the Roster in this respect, and will help to clarify the situation.

In conclusion I wish to recommend that problems of the use of scientists on the Roster be taken up with the Roster whenever they arise. With the advice of consultants in the various fields, the Roster is the group best organized and equipped to handle these questions.

December 26, 1941

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